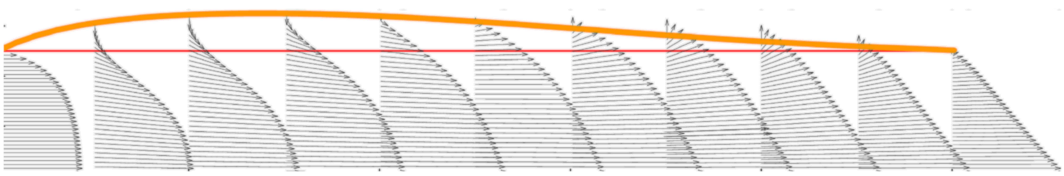
Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

2D Navier-Stokes Equations in a Time-Dependent Domain

Dissertation Thesis



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2D Navier-Stokes Equations in a Time-Dependent Domain

A DISSERTATION
SUBMITTED TO THE FACULTY
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Anna Zaušková

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It is the glory of God to conceal a thing;
but the honour of kings is
to search out the matter...

Thanks to all who encouraged me during my work. I greatly thank to my supervisor Ján Filo. This thesis would have hardly be written without him. His precise scientific reasoning and willingness to explain things over and over led me through the most difficult passages of our research.

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Contents

Preface

We study Navier-Stokes equations which model the non-stationary incompressible flow of the Newtonian fluid. The motivation for our study was a problem, where the geometry of the flow domain changes in time according to fluid properties such as stress tensor. After choosing an appropriate mathematical model of the flow in a domain with viscoelastic or compliant walls, we deal with its theoretical analysis, together with some numerical analysis and experiments. The motivation for our study comes from medicine—the simulation of blood flow in arteries and veins. The field of hemodynamics and fluid-structure interaction has been intensively studied world-wide during the last years, see e.g. proceedings of IUTAM Symposium on Flow past Highly Compliant Boundaries and in Collapsible Tubes [CP03].

One application of the study of physiological behavior of the vessel wall is a prediction of the stenosis, inner lumen (radius) restriction of a vessel which is a consequence of fat accumulation in vessels. For instance, it is quite usual to observe a partial reversal of the flow during the cardiac cycle in the area of carotid bifurcation. There is an evidence that one of the factors which causes fat accumulation is related to the oscillatory nature of the vessel wall stresses induced by the fluid in the flow reversal zone. Sometimes it is simpler to compute this stress and the flow field than to make measurements in vivo on a patient.

Another application of computational simulation of the flow is a prediction of the flow behavior after a modification of the geometry by a surgical operation such as bypass. Numerical simulations can determine the best bypass configuration. The wall deformation of the vessel between systole and diastole can achieve even 10% of its radius. This is the reason, mainly by large vessels, why elasticity of the wall must be considered in the mathematical model. In this case, the mathematical model also includes interaction between geometry of vessel wall and fluid flow. Such an interaction appears also in other biomedical applications, e.g. the hearth valve [DVV03].

The main goal of this work is to contribute to this interesting research area from the mathematical theory point of view, as well as by computational experiments. The problem of the blood flow is complex, e.g. modeling of vessels involves a complex 3D geometry of pipe system. In order to achieve some theoretical results, we consider only a very simple 2D geometry. Our work is based mainly on the blood flow model proposed by Alfio Quarteroni, see e.g. [FGNQ00], [Qua02], [ČLMT05]. Since we were not able to prove the existence and uniqueness for Quarteroni's coupled fluid-structure problem, we add an ε-regularisation to the continuity equation div $\boldsymbol{u} = 0$. We also add an approximation parameter κ in order to split the fluid-domain interface condition. Moreover, we used a method for decoupling of the fluid flow and the domain which differs from the methods in [FGNQ00] or [DDFQ06] and we refer to it as *global method*. In the global method, the fluid domain is described by an a priori given deformation $h(x, t)$ of its geometry for the whole time interval $t \in (0, T)$ and $x \in \partial\Omega$. Once the domain deformation is known, we prove the existence, uniqueness and continuous dependence on data. After passing the parameters to the limit, $\varepsilon \to 0$, $\kappa \to \infty$, we obtain the original Quartetoni's model. Also in numerical experiments, we first compute the initial velocity field, pressure and deformation in a domain deformed according to a given deformation $h(x, t)$ for $t \in (0, T)$. In the next iteration, we update the domain geometry using the most recent values of velocity and pressure (more precisely, the new values of the deformation are computed from the deformation equation with the fluid stress tensor on the right-hand side) and then compute the fluid flow for the updated geometry. We study the experimental convergence of this method in spite of the fact that we do not prove the convergence theoretically.

In Chapter 1, we introduce the mathematical model with approximated fluid-domain interface condition, studied in this thesis. In Chapter 2, we present a sequence of models which leads to a regularised system of equations with an approximated fluid-domain interface condition. We begin with the original Quarteroni's model and we continue with a derivation of the equation for the domain deformation in radially symmetric case, as it is explained in [Qua01]. Although we perform numerical experiments using this original model, we could not analytically prove the existence of the solution. Therefore we regularise Quarteroni's model. In order to do so, we replace the incompressibility condition for the fluid (i.e., velocity is divergence-free) with a parabolic equation for pressure. Moreover, we introduce new terms into both the momentum equation and the boundary condition. Most importantly, we approximate the condition for domain deformation and velocity on the elastic part of the boundary. This approximation introduces another right-hand side term for the deformation equation, as well as a different boundary condition on the elastic wall. In this step, we also outline the way of decoupling the fluid and structure equation. Furthermore, we use a Neumann-type boundary condition for fluid on the elastic part of the boundary and finish this chapter with introducing the final mathematical model, for which we prove the existence and uniqueness.

In Chapter 3, we introduce some notation and assumptions used in the following chapters. For the purpose of a theoretical analysis of our final model, we transform Navier-Stokes equations from time-dependent domain to a fixed rectangular domain. We also define a weak solution to the problem introduced in Chapter 2.

In Chapter 4, we prove the existence and uniqueness of the weak solution, which is the main theoretical result of this work. The method of the proof is similar to the method which is used in [Fei93]. The proof of existence is based on the implicit time discretisation, proving existence of the solution to the stationary problem, then proving a priori estimates and finally proving the convergence of piecewise-constant or piecewise-linear functions to the weak solution as time steps go to zero. The proof of convergence is based on the compactness of the solution in the appropriate function space $(L^p(D \times (0,T))^2$ for any $p < 4)$. In this crucial point of our approach, we follow the idea from [AL83] which differs from the method used in [Tem79] in the way of proving the compactness. We also prove existence of the functional $\frac{\partial(\boldsymbol{u})}{\partial t} \in X^* = L^2(0,T;\boldsymbol{V}^*) + L^{4/3}(0,T;L^{4/3}(D)^2)$. We extend the result of Feistauer [Fei93] by proving more general uniqueness for the regularised model. Our generalised uniqueness property depends on data (the data describe e.g. the deformation of the domain or the pressure on the boundary). We prove that if two data-sets are sufficiently close to each other, then two corresponding solutions to our problem are also close to each other.

Chapter 5 is devoted to numerical experiments with Quarteroni's model and with our model (Quarteroni's fluid-domain interface condition is split in our model). An implementation of the time and space discretisation for Navier-Stokes system is a standard part of the UG software package. A UG application which implements the moving-boundary problem had already been implemented by Philip Julian Broser [Bro02]. We extended Broser's work with an implementation of a wall-deformation equation solver which includes discretisation. We describe both the standard UG discretisation of Navier-Stokes system and the time and space discretisation of the deformation equation. We also describe the numerical realisation of the global method, i.e. the algorithm of decoupling the fluid (flow) and structure (domain) interaction based on iteration with respect to the domain. We finally present numerical experiments with the original Quarteroni's model and with our model using the global iterative method.

Chapter 1

Introduction

This work deals with the two-dimensional Navier-Stokes system for incompressible fluid

$$\rho\frac{\partial v_i}{\partial t} + \rho\sum_{i=1}^{2} v_j\frac{\partial v_i}{\partial x_j} = \mu\Delta v_i - \frac{\partial p}{\partial x_i}, \qquad i = 1, 2,$$
$$\operatorname{div} \boldsymbol{v} = 0 \tag{1.1}$$

in a time-dependent domain

$$\Omega(h) \equiv \{(x_1, x_2, t) : \ 0 < x_1 < L, \ 0 < x_2 < h(x_1, t), \ 0 < t < T\}$$

given by a known function of the domain deformation h, $h \in W^{2,2}((0,T) \times (0,L)) \cap C^1([0,T] \times [0,L])$ satisfying

$$0 < \alpha \le h(x_1, t) \le \alpha^{-1} , \quad h(0,t) = h(L,t) = h(x_1, 0) = \ell > 0. \tag{1.2}$$

On the upper part of the boundary of $\Omega(h)$, which is viscoelastic and deforms and which we shall denote Γ^w, we impose the following Neumann-type boundary condition for the second component of the velocity $\boldsymbol{v}$

$$\left[\mu\frac{\partial v_2}{\partial x_1}\left(-\frac{\partial h}{\partial x_1}\right) + \mu\frac{\partial v_2}{\partial x_2} - p + P_w - \frac{\rho}{2}v_2\left(v_2 - \frac{\partial h}{\partial t}\right)\right](x_1, h(x_1,t), t)$$
$$= \rho\kappa\left(\lambda\frac{\partial \eta}{\partial t}(x_1,t) + (1-\lambda)\frac{\partial h}{\partial t}(x_1,t) - v_2(x_1, h(x_1,t), t)\right) \tag{1.3}$$

for a given function $P_w = P_w(x_1, t)$, $0 < \lambda \le 1$, some $\kappa \gg 1$ and constant ρ. In this boundary condition, an unknown function $\eta = \eta(x_1, t)$ appears. We require η to satisfy the following differential equation

$$-E\left[\frac{\partial^2 \eta}{\partial t^2} - a\frac{\partial^2 \eta}{\partial x_1^2} + b\eta - c\frac{\partial^3 \eta}{\partial t \partial x_1^2}\right](x_1, t)$$
$$= \kappa\left(\lambda\frac{\partial \eta}{\partial t}(x_1,t) + (1-\lambda)\frac{\partial h}{\partial t}(x_1,t) - v_2(x_1, h(x_1,t), t)\right) \tag{1.4}$$

for any $0 < x_1 < L$, $0 < t < T$ which is equipped with the boundary and initial conditions for η

$$\eta(0,t) = \eta(L,t) = 0 \quad \text{and} \quad \eta(x_1,0) = \frac{\partial \eta}{\partial t}(x_1,0) = 0. \tag{1.5}$$

Moreover, we require at this part of boundary that $v_1 = 0$, i.e.

$$v_1(x_1, h(x_1,t), t) = 0 \qquad \text{for} \quad 0 < x_1 < L,\ 0 < t < T. \tag{1.6}$$

Hereby, E, a, b, c are given positive constants which will be explained later.

In accordance with our motivation described in Chapter 2, we complete the Navier-Stokes system (1.1) using the following boundary and initial conditions. On the inflow part of the boundary, which we shall denote Γ^{in}, we set

$$v_2(0,x_2,t) = 0, \qquad \left(\mu \frac{\partial v_1}{\partial x_1} - p + P_{in} - \frac{\rho}{2}|v_1|^2\right)(0,x_2,t) = 0 \tag{1.7}$$

for any $0 < x_2 < 1$, $0 < t < T$ and for a given function $P_{in} = P_{in}(x_2,t)$. On the opposite, outflow part of the boundary Γ^{out}, we set

$$v_2(L,x_2,t) = 0\ , \qquad \left(\mu \frac{\partial v_1}{\partial x_1} - p + P_{out} - \frac{\rho}{2}|v_1|^2\right)(L,x_2,t) = 0 \tag{1.8}$$

for any $0 < x_2 < 1$, $0 < t < T$ and for a given function $P_{out} = P_{out}(x_2,t)$. Finally, on the remaining part of the boundary, Γ^c, we set the flow symmetry condition

$$v_2(x_1,0,t) = 0\ , \qquad \mu \frac{\partial v_1}{\partial x_2}(x_1,0,t) = 0 \tag{1.9}$$

for any $0 < x_1 < L$, $0 < t < T$ and

$$\boldsymbol{v}(x_1,x_2,0) = \mathbf{0} \qquad \text{for any } 0 < x_1 < L,\ 0 < x_2 < h(x_1,0). \tag{1.10}$$

Chapter 2

Motivation

The problem described by the set of equations (1.1)–(1.5) is an approximation of the fluid-structure interaction model proposed by A. Quarteroni e.g. in [Qua02], [Qua01]. This model, which represents the coupled *fluid-structure* problem, is described in Section 2.1. The fluid flow problem is represented by system of Navier-Stokes equations for incompressible fluid, the structure-problem includes a wall deformation of an elastic tube. Section 2.2 presents a derivation of the mathematical model for a vessel wall. Our regularisation of this *fluid-structure* problem yields a new approximation of the problem which we discuss in Section 2.3. Our approximation also takes into account the decoupling of the fluid-structure interaction.

2.1 Original model

In this part we use the notation from [Qua01] and focus on the problem in 2D or 3D. For a Newtonian incompressible fluid in an elastic tube, we study a system of coupled equations for unknown velocity $\boldsymbol{v}$, pressure p and domain displacement η. The problem is defined in a tube of length L, with reference radius R_0 (see Fig. 2.1). The wall of the tube is deformed in radial direction by a deformation $\eta(x_1, t)$. We consider a time-dependent domain $\Omega_t \in \mathbb{R}^n,\ n = 2, 3$

$$\Omega_t = \{(x_1, x'),\ |x'| < R_0 + \eta(x_1, t),\ x_1 \in (0, L)\},\ t \in I,$$

where $I = (0, T)$ is the time interval, $x' = (x_2, \ldots, x_n)$ with boundaries

$$\begin{aligned} \Gamma_t^{wall} &= \{(x_1, x'),\ |x'| = R_0 + \eta(x_1, t), x_1 \in (0, L)\},\ t \in I, \\ \Gamma_t^{in} &= \{(0, x'),\ |x'| < R_0 + \alpha(t)\},\ t \in I, \\ \Gamma_t^{out} &= \{(L, x'),\ |x'| < R_0 + \beta(t)\},\ t \in I. \end{aligned}$$

The unknown quantities, all defined on Ω_t are: velocity field $\boldsymbol{v}(x, t) : \Omega_t \times I \to \mathbb{R}^n, n = 2, 3$, fluid pressure $p(x, t) : \Omega_t \times I \to \mathbb{R}$ and domain deformation $\eta(x_1, t)$:

$(0, L) \times I \to \mathbb{R}$, described by the following equations

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nu \text{ div } (\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T) + \nabla \bar{p} = \mathbf{f} \text{ in } \Omega_t, \ t \in I, \tag{2.1}$$

$$\text{div } \boldsymbol{v} = 0 \text{ in } \Omega_t, \ t \in I,$$

$$\frac{\partial^2 \eta}{\partial t^2} - a\frac{\partial^2 \eta}{\partial x_1^2} + b\eta - c\frac{\partial^3 \eta}{\partial t \partial x_1^2} = H, \ x_1 \in (0, L), \ t \in I, \tag{2.2}$$

with the following boundary conditions

$$\boldsymbol{v}(x, 0) = \boldsymbol{v}_0(x), \ x \in \Omega_0, \tag{2.3}$$

$$\eta(x_1, 0) = \eta_0(x_1), \ \frac{\partial \eta}{\partial t}(x_1, 0) = \eta_1(x_1), \ x_1 \in [0, L], \tag{2.4}$$

$$\big(\nu(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T) - (\bar{p} - \bar{P}_{out})\mathbf{I}\big)\boldsymbol{n} = 0 \text{ on } \Gamma_t^{out}, \ t \in I, \tag{2.5}$$

$$\boldsymbol{v} = \mathbf{g}(x, t), \text{ or} \tag{2.6}$$

$$\big(\nu(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T) - (\bar{p} - \bar{P}_{in}(t))\mathbf{I}\big)\boldsymbol{n} = 0 \text{ on } \Gamma_t^{in}, \ t \in I,$$

$$\eta(0, t) = \alpha(t) \quad \eta(L, t) = \beta(t), \ t \in I, \tag{2.7}$$

and the following condition on the fluid-structure interface

$$\boldsymbol{v} = \frac{\partial \eta}{\partial t}\boldsymbol{n} \quad \text{on } \Gamma_t^{wall} \quad t \in I, \tag{2.8}$$

here $\bar{p} = \frac{p}{\rho}$, $\bar{P}_{in}(t) = \frac{P_{in}(t)}{\rho}$, $\bar{P}_{out}(t) = \frac{P_{out}(t)}{\rho}$, $\nu = \frac{\mu}{\rho}$. Functions $\mathbf{f}(x, t)$, $\boldsymbol{v}_0(x)$, $\eta_0(x_1)$, $\eta_1(x_1)$, $\mathbf{g}(x, t)$, $P_{in}(t)$, $P_{out}(t)$ are given. The function $P_{in}(t)$ is the inflow pressure, P_{out} is the outflow pressure and μ is the viscosity, where we consider ρ a constant density of the fluid.

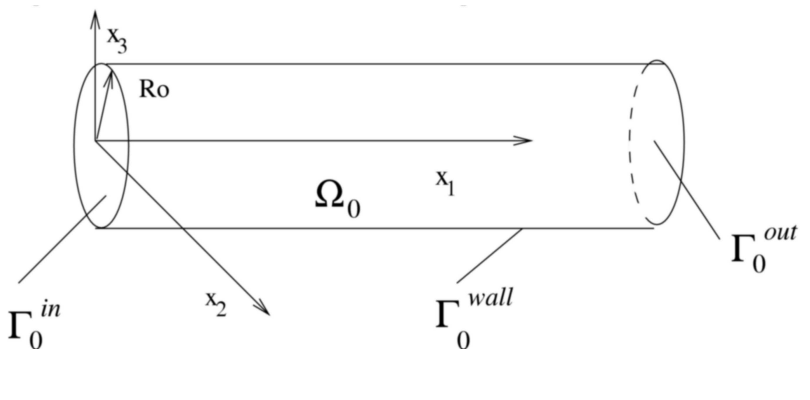


Figure 2.1: Reference domain in $t = 0$

The coupling between the momentum equation for the flow (2.1) and the wall deformation equation (2.2) is twofold. First, the source term of (2.2) contains the solution to Navier-Stokes equations (in the form of Cauchy stress tensor)

$$H = \frac{1}{\rho_w h}\Big(p - P_w + \nu\big((\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)\cdot \boldsymbol{n}\big)\cdot \boldsymbol{e}_r\Big),$$

where ρ_w, h are given constants, and P_w is an external tissue pressure acting on the deformed wall, $\boldsymbol{n}$ is the external normal, and $\boldsymbol{e}_r = (0, \frac{x_2}{|x'|}, \dots, \frac{x_n}{|x'|})$ is the vector of the radial direction.

The second coupling phenomenon is the non-homogeneous Dirichlet boundary condition on the deformed wall (2.8), which ensures that the velocity of the wall movement is equal to the fluid velocity on the tube wall. The interaction between fluid and tube wall means that the fluid solution provides the forcing term required by the deformation equation (2.2). On the other hand, the movement of the wall changes the geometry on which the fluid equations are to be solved, and also modifies the Dirichlet boundary condition (2.8). An iterative algorithm which decouples this system was introduced in [Qua01] and is described in more detail in [FGNQ00]. Other fluid-structure decoupling algorithms can be found in [DDFQ06].

2.2 Derivation of the domain displacement equation

The detailed derivation of a *generalised string model* (2.2) for domain deformation is given in [Qua01]. For more complicated, so-called shell models, which are not covered in this thesis, see the book of P. G. Ciarlet [Cia98]. Other models for the elastic wall, together with physical characteristics of elastic materials, can be found in [CP03], [Mil89]. In the sequel, a three-dimensional flow problem in a cylindrical domain is considered, where the deformation function is expressed in radial coordinates. We give a brief overview of its derivation.

Definition 2.1 (Domain deformation)**.** *The domain deformation is defined as a function dependent on angle θ, longitudinal variable z and time t:*

$$\eta(\theta, z, t) \equiv R(\theta, \boldsymbol{z}, t) - R_0,$$

where R_0 is the reference radius of a tube and $R(\theta, z, t)$ is the actual radius.

We make the following assumptions in our model:

A1 *Cylindrical domain.* The reference geometry is a cylinder with no branches, see Fig. 2.2.

A2 *The thickness of the wall is a small constant h.**

*In this section, h denotes the constant thickness of the wall. In the following sections, $h(x_1, t)$ denotes a known function which describes the domain deformation and the constant h is hidden in coefficients a, b, c of the deformation equation for η (1.4).

A3 *The deformation gradient is small with respect to* z, θ. This means that $\frac{\partial \eta}{\partial z}$, $\frac{\partial \eta}{\partial \theta}$ are small.

A4 *Normal stresses.* The surface stresses σ_z, σ_θ are directed along the normal to the surface $z = const,\ \theta = const$ to which the stresses apply, as it is indicated in Fig. 2.3.

A5 $|\sigma_\theta|$ *is constant with respect to* θ*.*

A6 *Wall displacements are only applied in radial directions.*

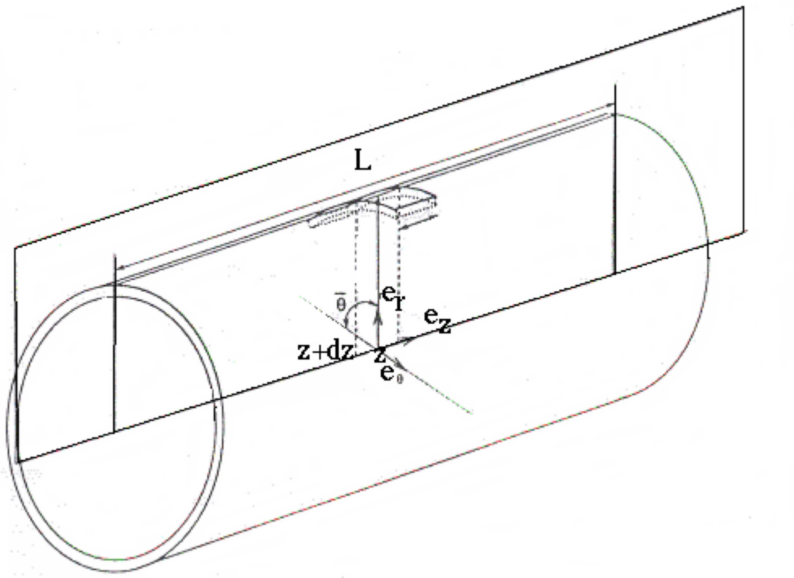


Figure 2.2: A cylindrical model of the reference geometry

Here are a few direct consequences of the assumptions made so far:

- From A1 follows that the normal can be approximated by a vector of radial direction $\boldsymbol{n} \approx -\boldsymbol{e}_r$, the length of the arc $dc \approx R\ d\theta$, see Fig. 2.4.
- From assumptions A2 and A4 follows that the stress σ_θ is constant with respect to the angle, $\frac{\sigma_\theta}{\partial r} = 0$.
- A3 implies a linear elasticity of the vessel wall and also $dl \approx dz$, see Fig. 2.4.
- If A6 holds, then the model reduces to a differential equation for the wall deformation η for any fixed value of θ—all directions except radial ones are neglected.

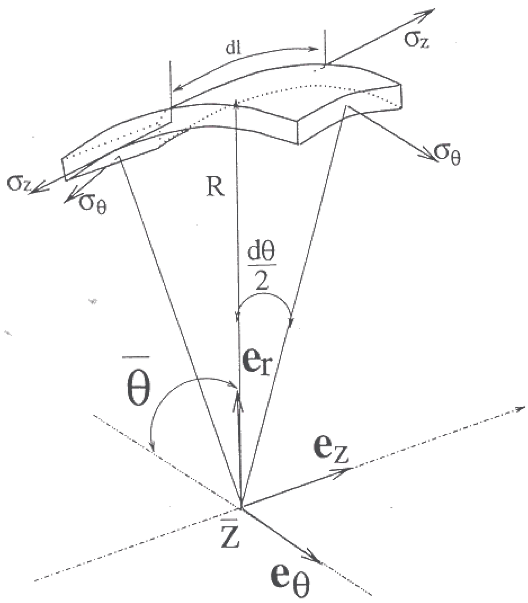


Figure 2.3: Small part of a vessel with physical characteristics, used for the derivation of the model

The direction of the longitudinal stress $\boldsymbol{\sigma}_z$, (Fig. 2.4) is at any time parallel to the tangent $\boldsymbol{\tau}$ of the curve $R(\bar{\theta}, z; t)$, which represents the geometry of the domain for the given $\theta = \bar{\theta}$. This means, if $\sigma_z = const > 0$ then $\boldsymbol{\sigma}_z = \pm\sigma_z\boldsymbol{\tau}$. The circumferential stress $\boldsymbol{\sigma}_\theta$ is oriented along the direction of the external normal of the surface to which the stress applies, thus $\sigma_\theta = \boldsymbol{\sigma}_\theta \cdot \boldsymbol{n}_\theta > 0$. Both stresses $\boldsymbol{\sigma}_z$, $\boldsymbol{\sigma}_\theta$ represent internal forces acting on the vessel portion. We consider the following forces acting on the vessel wall:

- *Forces from the surrounding tissues*, given by external pressure acting on the wall $P_w = P_{wall}$

$$f_{tissue} = \boldsymbol{f}_{tissue} \cdot \boldsymbol{e}_r = -P_w \ dc \ dl + o(dc \ dl) = -P_w R \ d\theta \ dl + o(d\theta \ dl).$$

  where o(y) is such that $lim_{y\to 0}\frac{o(y)}{y} = 0$.

- *Forces from the fluid*, represented by stress tensor $\boldsymbol{T}_f = -p + \mu(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T)$

$$f_{fluid} = (\boldsymbol{T}_f \cdot \boldsymbol{n}) \cdot \boldsymbol{e}_r \ dc \ dl + o(dc \ dl) = (\boldsymbol{T}_f \cdot \boldsymbol{n}) \cdot \boldsymbol{e}_r R \ d\theta \ dl + o(dc \ dl).$$

- *Internal forces*, stresses $\boldsymbol{\sigma}_\theta$, $\boldsymbol{\sigma}_z$ (we consider only their radial part)

$$f_{int} = \boldsymbol{f}_{int} \cdot \boldsymbol{e}_r = (\boldsymbol{f}_z + \boldsymbol{f}_\theta) \cdot \boldsymbol{e}_r,$$

where

$$\begin{aligned}
\boldsymbol{f}_\theta &= \Big[\boldsymbol{\sigma}_\theta\Big(\bar{\theta}+\frac{d\theta}{2}\Big)+\boldsymbol{\sigma}_\theta\Big(\bar{\theta}-\frac{d\theta}{2}\Big)\Big]h\ dl = \\
&= 2\sigma_\theta cos\Big(\frac{\pi}{2}+\frac{d\theta}{2}\Big)h\ dl = -2\sigma_\theta sin\frac{d\theta}{2}h\ dl = -\sigma_\theta h\ d\theta\ dl + o(d\theta\ dl), \\
\boldsymbol{f}_z &= \Big[\boldsymbol{\sigma}_z\Big(z^*+\frac{dz}{2}\Big)+\boldsymbol{\sigma}_z\Big(z^*-\frac{dz}{2}\Big)\Big]h\ dc = \\
&= \frac{\sigma_z\boldsymbol{\tau}(z^*+\frac{dz}{2})+\sigma_z\boldsymbol{\tau}(z^*-\frac{dz}{2})}{dz}h\ dz\ dc = \sigma_z\Big[\frac{d\boldsymbol{\tau}}{dz}(z^*)dz+o(dz)\Big]h\ dc
\end{aligned}$$

(h denotes the thickness of the vessel wall).

If $\frac{\partial R}{\partial z}$ is small enough, then we can assume that [Qua01, Lemma C.1]

$$\frac{d\boldsymbol{\tau}}{dz} \approx -\frac{\partial^2 R}{\partial z^2}\boldsymbol{n}.$$

For the radial part of $\boldsymbol{f}_z$, assuming $\boldsymbol{n} \approx -\boldsymbol{e}_r$ we obtain

$$f_z = \boldsymbol{f}_z \cdot \boldsymbol{e}_r \approx \sigma_z\frac{\partial^2 R}{\partial z^2}h\ dc\ dz \approx \sigma_z\frac{\partial^2 \eta}{\partial z^2}Rh\ d\theta\ dl.$$

For the internal stress $\boldsymbol{\sigma}_\theta$, we assume a linear elasticity, i.e. we assume that $\boldsymbol{\sigma}_\theta$ is proportional to the relative circumferential elongation:

$$\sigma_\theta = \mathcal{E}\frac{2\pi(R-R_0)}{2\pi R_0} = \frac{\mathcal{E}\eta}{R_0},$$

where $\mathcal{E}$ is Young's modulus of elasticity. Appropriate values of $\mathcal{E}$ for arteries can be found e.g. in [Mil89].

Finally, we use Newton's law, $F = m \cdot a$. For the mass of the wall portion and for the acceleration along the radial direction, it holds

$$m = \rho_w h\ dc\ dl = \rho_w hR\ d\theta\ dl,\ a = \frac{\partial^2 R}{\partial t^2} = \frac{\partial^2 \eta}{\partial t^2},$$

where ρ_w is density of the vessel wall and h is its thickness. Putting the mentioned forces into balance, using the Newton's law and neglecting higher order terms yields

$$\Big[\rho_w hR\frac{\partial^2\eta}{\partial t^2} - \sigma_z h\frac{\partial^2\eta}{\partial z^2}R - (\boldsymbol{T}_f\cdot\boldsymbol{n})\cdot\boldsymbol{e}_r R + P_w R + \mathcal{E}h\frac{\eta}{R_0}\Big]\ d\theta\ dl = 0.$$

For a small domain deformation, we can assume $R \approx R_0$. Division of the last equation by radius R yields an equation called a *vibrating string model*:

$$\frac{\partial^2\eta}{\partial t^2} - \frac{\sigma_z h}{\rho_w h}\frac{\partial^2\eta}{\partial z^2} + \frac{h\mathcal{E}}{\rho_w h R_0^2}\eta = \frac{p - P_w + (\mu(\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^T)\cdot\boldsymbol{n})\cdot\boldsymbol{e}_r}{\rho_w h}.$$

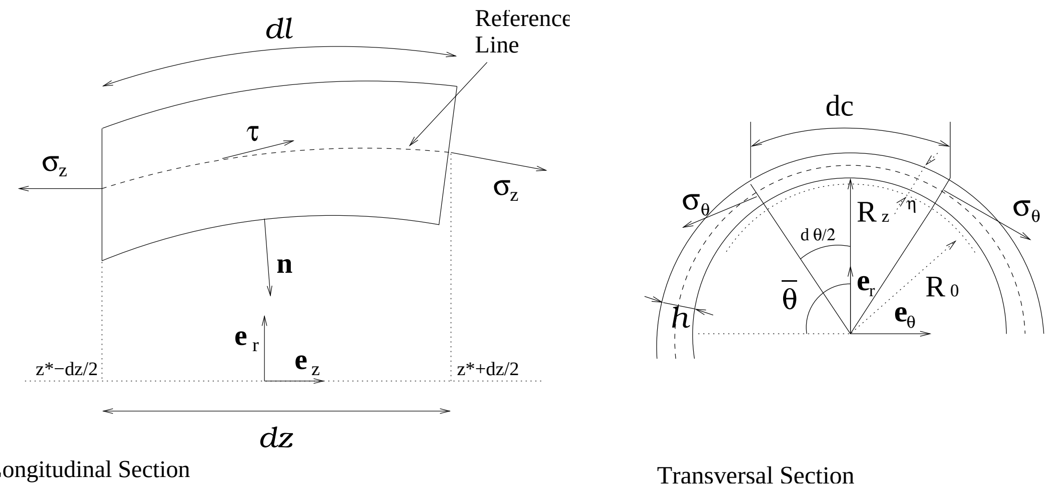


Figure 2.4: Physical quantities on longitudinal and transversal sections

Adding a damping term $-c\frac{\partial^3\eta}{\partial t\partial z^2}$ to the left-hand side of last expression, where c is positive constant, yields a *generalised string model* for the wall displacement:

$$\frac{\partial^2\eta}{\partial t^2} - \frac{\sigma_z h}{\rho_w h}\frac{\partial^2\eta}{\partial z^2} + \frac{h\mathcal{E}}{\rho_w h R_0^2}\eta - c\frac{\partial^3\eta}{\partial t\partial z^2} = \frac{p - P_w - (\mu(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)\cdot \boldsymbol{n})\cdot \boldsymbol{e}_r}{\rho_w h}.$$

Appropriate values of the physical constants which appear in this model can be found in [ČLMT05].

For longitudinal stress σ_z, we use [ČLMT05]:

$$\sigma_z = \kappa G,$$

where $\kappa = 1$ is Timoshenko's correction factor and G is shear modulus, $G = \frac{\mathcal{E}}{2(1+\gamma)}, \gamma = 0.5$ for an incompressible medium. These values are also used in our numerical experiments.

2.2.1 Original model with one-dimensional domain deformation

We restrict our research to a two-dimensional flow problem. This implies a one-dimensional model for domain deformation. The structure will be then represented as a line, see Fig. 2.5. This simplification is also used in [Qua01].

In case of 2D flow and 1D structure, the longitudinal direction z is denoted as x_1. We replace the radial coordinate r with x_2,

$$x_1 \equiv z, \qquad x_2 \equiv r, \qquad \boldsymbol{e}_r \equiv \boldsymbol{e}_2,$$

and the angle does not play any role in the structure problem.

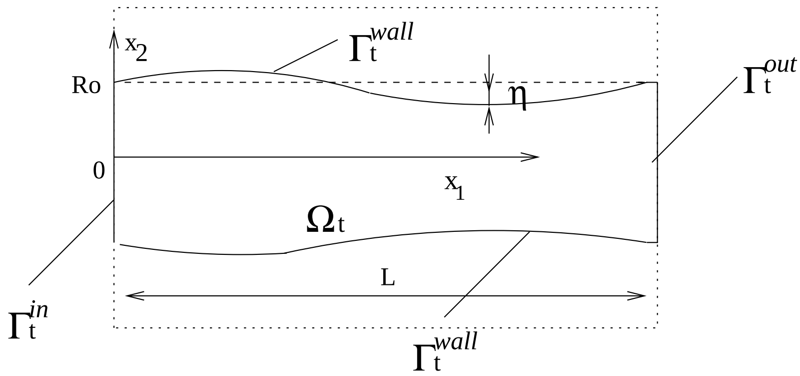


Figure 2.5: 2D time-deforming domain in Quarteroni's model

Before introducing the main idea of our approximation, we summarise the problem to which we refer as to the *original model* of A. Quarteroni, see e.g. [Qua02], [FGNQ00]. In the time-dependent domain with reference radius $R_0 \equiv \ell$ (see Fig. 2.6)

$$\Omega(\eta) \equiv \{(x_1, x_2, t) : \ 0 < x_2 < R_0 + \eta(x_1, t), \ 0 < x_1 < L, \ 0 < t < T\},$$

we are looking for a function

$$v_1 = v_1(x_1, x_2, t), \ v_2 = v_2(x_1, x_2, t), \ p = p(x_1, x_2, t) \text{ and } \eta = \eta(x_1, t)$$

with the following properties:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \text{div } \left(\mu(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)\right) - \nabla p, \ \text{div } \boldsymbol{v} = 0 \tag{2.9}$$

in $\Omega(\eta)$,

$$\frac{\partial^2 \eta}{\partial t^2} - a\frac{\partial^2 \eta}{\partial x_1^2} + b\eta - c\frac{\partial^3 \eta}{\partial t \partial x_1^2} = H \tag{2.10}$$

for any $0 < x_1 < L$, $0 < t < T$, where

$$H = \frac{1}{E\rho}\left(p - P_w - \mu\left((\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T) \cdot \boldsymbol{n}\right) \cdot \boldsymbol{e}_2\right), \ \boldsymbol{n} = \left(-\frac{\partial \eta}{\partial x_1}, 1\right), \tag{2.11}$$

and

$$\boldsymbol{v}(x_1, \ell + \eta(x_1, t), t) = \frac{\partial \eta}{\partial t} \frac{\boldsymbol{n}}{|\boldsymbol{n}|}(x_1, t). \tag{2.12}$$

The problem (2.9)–(2.12) is in [Qua01] equipped with the following boundary

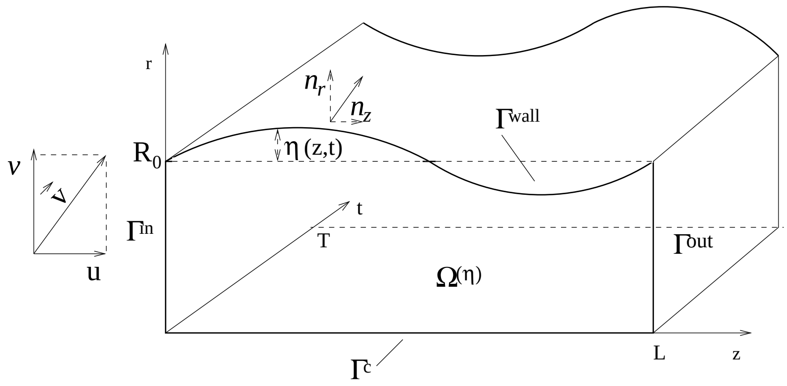


Figure 2.6: 2D time-dependent domain in our model

and initial conditions:

$$\eta(0,t) = \alpha(t), \quad \eta(L,t) = \beta(t), \quad t \in (0,T), \tag{2.13}$$

$$\eta(x_1,0) = \eta_0(x_1), \quad \frac{\partial \eta}{\partial t}(x_1,0) = \eta_1(x_1), \quad x_1 \in (0,L), \tag{2.14}$$

$$\boldsymbol{v} = \boldsymbol{f} \quad \text{on} \quad \Gamma^{in}, \tag{2.15}$$

$$\mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) = 0, \quad 2\mu\frac{\partial v_1}{\partial x_1} - p + P_{out} = 0 \quad \text{on} \quad \Gamma^{out}, \tag{2.16}$$

$$v_2 = 0, \quad \mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) = 0 \quad \text{on} \quad \Gamma^c, \tag{2.17}$$

$$\boldsymbol{v}(x_1,x_2,0) = \boldsymbol{v}_0(x_1,x_2) \quad x_1 \in (0,L), x_2 \in (0, \ell + \eta(x_1,0)) \tag{2.18}$$

for given functions α, β, η_0, η_1, $\boldsymbol{f}$, P_{out} and $\boldsymbol{v}_0$.

2.3 Our approximation of the original problem

In this section, we simplify and regularise the original problem (2.9)–(2.18) in several steps in order to end up with approximation (1.1)–(1.10) from Chapter 1.

We first replace the operator div $\big(\mu(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T)\big)$ from (2.9) with $\mu\Delta\boldsymbol{v}$. This simplification of the momentum equation we obtain after commuting the space derivations of $\boldsymbol{v}$ in operator div $\big(\mu(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T)\big)$ and by using div $\boldsymbol{v} = 0$. We also correspondingly modify the Neumannn-type boundary conditions and H from (2.10):

$$H = \frac{1}{\rho E}\left(p - P_w - \mu\frac{\partial v_2}{\partial x_1}\left(-\frac{\partial \eta}{\partial x_1}\right) - \mu\frac{\partial v_2}{\partial x_2}\right). \tag{2.19}$$

Then we replace (2.12) with

$$v_1(x_1, \ell + \eta(x_1,t), t) = 0 \quad \text{and} \quad v_2(x_1, \ell + \eta(x_1,t), t) = \frac{\partial \eta}{\partial t}(x_1, t). \tag{2.20}$$

2.3.1 Decoupling of the fluid-domain interaction

In the first step, we split the problem by decoupling the domain geometry from equations which are to be solved in this domain. We will assume a known deformation function $\eta^{(k)}(x_1,t) \equiv h(x_1,t) - \ell$. In a given domain

$$\Omega^{(k)} \equiv \Omega(\eta^{(k)})$$

we look for a solution

$$(\boldsymbol{v}, p, \eta) = (\boldsymbol{v}^{(k+1)}, p^{(k+1)}, \eta^{(k+1)})$$

of the following problem:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \mu \Delta \boldsymbol{v} - \nabla p \qquad \text{in} \quad \Omega^{(k)} \tag{2.21}$$

and

$$-E\rho \left[\frac{\partial^2 \eta}{\partial t^2} - a \frac{\partial^2 \eta}{\partial x_1^2} + b\eta - c \frac{\partial^3 \eta}{\partial t \partial x_1^2} \right] = \tag{2.22}$$

$$\left[\mu \frac{\partial v_2}{\partial x_1} \left(- \frac{\partial \eta^{(k)}}{\partial x_1} \right) + \mu \frac{\partial v_2}{\partial x_2} - p + P_w \right] (x_1, \ell + \eta^{(k)}, t)$$

for any $0 < x_1 < L$, $0 < t < T$.

We performed numerical computations, where we iterated with respect to the domain (k). We refer to this iterative process as *global iterative method*. Although we were not able to show the convergence of $\eta^{(k)} \to \eta$ for $k \to \infty$, numerical experiments presented in the last chapter indicate that the domain deformation $\eta(x_1,t)$ stabilises after a few global iterations.

2.3.2 Boundary condition on the deformed wall

The following step of our approximation is also used in [FL99]. The key idea is a reformulation of the condition (2.2) and the second condition on the interface Γ^{wall} (2.8) or (2.20), respectively.

$$\begin{aligned} &\left[\mu \frac{\partial v_2}{\partial x_1} \left(- \frac{\partial \eta^{(k)}}{\partial x_1} \right) + \mu \frac{\partial v_2}{\partial x_2} - p + P_w \right. \\ &\qquad \left. - \frac{\rho}{2} v_2 \left(v_2 - \frac{\partial \eta^{(k)}}{\partial t} \right) \right] (x_1, \ell + \eta^{(k)}(x_1,t), t) \\ &= \rho\kappa \left(\lambda \frac{\partial \eta}{\partial t}(x_1,t) + (1-\lambda) \frac{\partial \eta^{(k)}}{\partial t}(x_1,t) - v_2(x_1, \ell + \eta^{(k)}(x_1,t), t) \right) \end{aligned} \tag{2.23}$$

We also replace (2.19) with

$$-E\rho\left[\frac{\partial^2\eta}{\partial t^2}-a\frac{\partial^2\eta}{\partial x_1^2}+b\eta-c\frac{\partial^3\eta}{\partial t\partial x_1^2}\right](x_1,t)$$
$$=\kappa\left(\lambda\frac{\partial\eta}{\partial t}(x_1,t)+(1-\lambda)\frac{\partial\eta^{(k)}}{\partial t}(x_1,t)-v_2(x_1,\ell+\eta^{(k)}(x_1,t),t)\right) \tag{2.24}$$

for $\kappa\gg 1,\ 0\le\lambda\le 1$. This is the crucial point of our approach. Note that if

$$\kappa\longrightarrow\infty$$

then—at least formally—the second equality of (2.20) holds for the known domain deformation (for $\lambda=1$), i.e. $v_2(x_1,\ell+\eta^{(k)}(x_1,t),t)=\frac{\partial\eta^{(k+1)}}{\partial t}(x_1,t)$, see Remark 4.1. This is a consequence of the first a priori estimate in Section 4.2. Moreover, the deformation equation is satisfied in the following manner:

$$\left[\mu\frac{\partial v_2}{\partial x_1}\left(-\frac{\partial h}{\partial x_1}\right)+\mu\frac{\partial v_2}{\partial x_2}-p+P_w\right](x_1,h,t)=$$
$$-E\rho\left[\frac{\partial^2\eta}{\partial t^2}-a\frac{\partial^2\eta}{\partial x_1^2}+b\eta-c\frac{\partial^3\eta}{\partial t\partial x_1^2}\right]$$

for any $0<x_1<L$, $0<t<T$, (i.e. 2.10 and 2.19 hold). Thus we arrive at the problem rather similar to the original, however, h is the known domain deformation, and η is to be found such that it satisfies the previous equation.

Moreover, note that we use the combination of the unknown and given deformation on the right-hand side of (2.23) and (2.24), and thus (again, see the a priori estimates later, for $0\le\lambda<1$), except of external pressures acting on the boundaries, we also obtain a new source for fluid flow in time-dependent domain: the domain deformation $h=\ell+\eta^{(k)}$.

2.3.3 Boundary and initial conditions

We simplify the boundary and initial conditions (2.13)–(2.14) for η by setting $\alpha(t)=\beta(t)\equiv 0$ and $\eta_0(x_1)=\eta_1(x_1)\equiv 0$. We replace the inflow condition (2.15) given by nonhomogeneous Dirichlet boundary condition with Neumann-type boundary conditions involving pressure from two reasons. It seems to be more natural to have given pressure impulses $P_{in}(\cdot,t)$ on Γ^{in} than prescribed velocity $\boldsymbol{f}$. Furthermore, we tried to avoid the problem of finding an extension $\boldsymbol{f}_{ext}$ of $\boldsymbol{f}$ given on Γ^{in} on the whole domain $\Omega^{(k)}$ such that

$$\operatorname{div}\boldsymbol{f}_{ext}=0 \qquad \text{in}\ \ \Omega^{(k)}.$$

As for the boundary conditions on Γ^{in} and Γ^{out}, we prescribe the second component of velocity $v_2 = 0$ in both cases and we use the Neumann-type boundary conditions for the first component of the velocity. This involves a use of the dynamic pressure

$$p + \frac{\rho}{2}\,|\boldsymbol{v}|^2 \qquad \text{instead of static pressure} \quad p.$$

This type of boundary conditions is studied in [HRT96], [QV97] (and it is mentioned e.g. in [Qua01] as well). The problem of 3D-flow in a network of pipes is studied in [CMP94], where boundary conditions involving the pressure

$$\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \qquad \text{and} \quad p + \frac{\rho}{2}\,|\boldsymbol{v}|^2 = p_0$$

are prescribed for Γ^{in}. The boundary conditions on Γ^c represent the assumptions on symmetry.

2.4 Additional regularisation

Finally, in order to overcome the difficulty with the incompressibility condition

$$\text{div}\ \boldsymbol{v} = 0 \qquad \text{in} \quad \Omega(h),$$

we drop this equation and replace the incompressibility condition with a parabolic equation for the pressure:

$$\varepsilon\left(\frac{\partial p}{\partial t} - \Delta p\right) + \text{div}\ \boldsymbol{v}_\varepsilon = 0 \qquad \text{in} \quad \Omega(h) \tag{2.25}$$

for a small $0 < \varepsilon < 1$. We follow [Tem79] in this step, where the following approximation is used

$$\varepsilon\frac{\partial p}{\partial t} + \text{div}\ \boldsymbol{v}_\varepsilon = 0.$$

Other strategies can be found e.g. in [QV97], [Bän98], where a similar regularisation is used in the operator-splitting method and corresponds to solving $\Delta p = \text{div}\ \boldsymbol{v}$ in each time step. The convergence of $\boldsymbol{v}_\varepsilon$ if $\varepsilon \to 0$ for (2.25) is also shown in Chapter 4.

The last step of the approximation of original problem is the addition of the term

$$\frac{\rho}{2} v_i\ \text{div}\ \boldsymbol{v}$$

into a momentum equation (2.9). This method is used also in the book of Temam [Tem79]. After all the previously described approximations and regularisations, we arrive at the problem mentioned in Chapter 1. For a given $\kappa \gg 1, \varepsilon \ll 1,\ 0 \leq \lambda \leq 1$ and a smooth function h, consider the system

$$\begin{aligned} \rho\frac{\partial v_i}{\partial t} + \rho\sum_{i=1}^{2} v_j\frac{\partial v_i}{\partial x_j} + \frac{\rho}{2}\, v_i \operatorname{div}\boldsymbol{v} &= \mu\Delta v_i - \frac{\partial p}{\partial x_i}, \qquad i = 1,2 \\ \varepsilon\left(\frac{\partial p}{\partial t} - \Delta p\right) + \operatorname{div}\boldsymbol{v} &= 0 \end{aligned} \tag{2.26}$$

in $\Omega(h)$,

$$\begin{aligned} &\left[\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x_1^2} + b\eta - c\frac{\partial^3\eta}{\partial t\partial x_1^2}\right](x_1,t) = \\ &-\frac{\kappa}{\rho E}\left(\lambda\frac{\partial\eta}{\partial t}(x_1,t) + (1-\lambda)\frac{\partial h}{\partial t}(x_1,t) - v_2(x_1,h(x_1,t),t)\right) \end{aligned} \tag{2.27}$$

for any $0 < x_1 < L,\ 0 < t < T$, equipped with the following boundary and initial conditions:

$$v_1(x_1,h(x_1,t),t) = 0$$

$$\begin{aligned} &\left[\mu\frac{\partial v_2}{\partial x_1}\left(-\frac{\partial h}{\partial x_1}\right) + \mu\frac{\partial v_2}{\partial x_2} - p + P_w - \frac{\rho}{2}v_2\left(v_2 - \frac{\partial h}{\partial t}\right)\right](x_1,h(x_1,t),t) \\ &\qquad = \kappa\left(\lambda\frac{\partial\eta}{\partial t}(x_1,t) + (1-\lambda)\frac{\partial h}{\partial t}(x_1,t) - v_2(x_1,h(x_1,t),t)\right) \\ &\left[\frac{\partial p}{\partial x_1}\left(-\frac{\partial h}{\partial x_1}\right) + \frac{\partial p}{\partial x_2}\right](x_1,h(x_1,t),t) = -\frac{\rho}{2}\frac{\partial h}{\partial t}(x_1,t)\ p(x_1,h(x_1,t),t) \end{aligned}$$

for any $0 < x_1 < L,\ 0 < t < T$,

$$\left(\mu\frac{\partial v_1}{\partial x_1} - p + P_{out} - \frac{\rho}{2}\,|\boldsymbol{v}|^2\right)(L,x_2,t) = 0$$

$$v_2(L,x_2,t) = 0, \qquad \frac{\partial p}{\partial x_1}(L,x_2,t) = 0$$

for any $0 < x_2 < \ell$, $0 < t < T$,

$$\left(\mu\frac{\partial v_1}{\partial x_1} - p + P_{in} - \frac{\rho}{2}\,|\boldsymbol{v}|^2\right)(0,x_2,t) = 0$$

$$v_2(0,x_2,t) = 0, \qquad \frac{\partial p}{\partial x_1}(0,x_2,t) = 0$$

for any $0 < x_2 < \ell$, $0 < t < T$,

$$\mu \frac{\partial v_1}{\partial x_2}(x_1, 0, t) = 0, \quad v_2(x_1, 0, t) = 0, \quad \frac{\partial p}{\partial x_2}(x_1, 0, t) = 0$$

for any $0 < x_1 < L$, $0 < t < T$,

$$\boldsymbol{v}(x_1, x_2, 0) = \mathbf{0}, \quad p(x_1, x_2, 0) = 0$$

for any $0 < x_2 < \ell$, $0 < x_1 < L$,

$$\eta(x_1, 0) = 0, \quad \frac{\partial \eta}{\partial t}(x_1, 0) = 0$$

for any $0 < x_1 < L$ and finally,

$$\eta(0, t) = \eta(L, t) = 0$$

for any $0 < t < T$.

Chapter 3

Formulation of our problem

In this and the following chapter we study the existence, uniqueness and regularity of solution to the problem (2.26)–(2.27). In Section 3.1, we transform variables in order to arrive at the problem on a fixed domain. A weak formulation of the problem and some interpolation inequalities frequently used in proofs are introduced in Section 3.2.

3.1 Transformation to the fixed domain

We first transform the final problem from Section 2.4 to the fixed domain. After tedious but straightforward manipulations, it can be shown that if $(\boldsymbol{v}, p, \eta)$ solves the problem (2.26)–(2.27), then $(\boldsymbol{u}, q, u)$ such that

$$\begin{aligned}
\boldsymbol{u}(y_1, y_2, t) &\stackrel{\text{def}}{=} \boldsymbol{v}(y_1, h(y_1, t)y_2, t) \\
q(y_1, y_2, t) &\stackrel{\text{def}}{=} \rho^{-1} p(y_1, h(y_1, t)y_2, t) \\
u(y_1, t) &\stackrel{\text{def}}{=} \frac{\partial \eta}{\partial t}(y_1, t)
\end{aligned}$$

for $0 < y_1 < L$, $0 < y_2 < 1$, $0 < t < T$ solves the following problem:

$$\begin{aligned}
&\frac{\partial (hu_1)}{\partial t} - \frac{\partial h}{\partial t}\frac{\partial (y_2 u_1)}{\partial y_2} + hu_1 \left(\frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2} \right) + u_2 \frac{\partial u_1}{\partial y_2} \\
&\quad + \frac{h}{2} u_1 \operatorname{div}_h \boldsymbol{u} - \frac{\partial}{\partial y_1}\left[\nu h \left(\frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2} \right) - hq \right] \\
&\quad - \frac{\partial}{\partial y_2}\left[\frac{\nu}{h}\frac{\partial u_1}{\partial y_2} - \nu y_2 \frac{\partial h}{\partial y_1}\left(\frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2} \right) + y_2 \frac{\partial h}{\partial y_1} q \right] = 0
\end{aligned} \tag{3.1}$$

in

$$D \times (0, T) \equiv \{(y_1, y_2) : \ 0 < y_1 < L, \ 0 < y_2 < 1\} \times \{t : 0 < t < T\},$$

$$\begin{aligned}\frac{\partial(hu_2)}{\partial t} - \frac{\partial h}{\partial t}\frac{\partial(y_2u_2)}{\partial y_2} + hu_1\left(\frac{\partial u_2}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_2}{\partial y_2}\right) + u_2\frac{\partial u_2}{\partial y_2} \\ + \frac{h}{2}u_2 \,\mathrm{div}_h\, \boldsymbol{u} - \frac{\partial}{\partial y_1}\left[\nu h\left(\frac{\partial u_2}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_2}{\partial y_2}\right)\right] \\ -\frac{\partial}{\partial y_2}\left[\frac{\nu}{h}\frac{\partial u_2}{\partial y_2} - \nu y_2\frac{\partial h}{\partial y_1}\left(\frac{\partial u_2}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_2}{\partial y_2}\right) - q\right] = 0\end{aligned} \tag{3.2}$$

in $D \times (0, T)$,

$$\begin{aligned}\varepsilon\left(\frac{\partial(hq)}{\partial t} - \frac{\partial h}{\partial t}\frac{\partial(y_2q)}{\partial y_2}\right) - \varepsilon\frac{\partial}{\partial y_1}\left[h\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right] \\ -\varepsilon\frac{\partial}{\partial y_2}\left[\frac{1}{h}\frac{\partial q}{\partial y_2} - y_2\frac{\partial h}{\partial y_1}\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right] + \ h\,\mathrm{div}_h\,\boldsymbol{u} = 0\end{aligned} \tag{3.3}$$

in $D \times (0, T)$, where

$$\mathrm{div}_h\,\boldsymbol{u} \ \stackrel{\mathrm{def}}{=}\ \frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2} + \frac{1}{h}\frac{\partial u_2}{\partial y_2} \tag{3.4}$$

$$\begin{aligned}&\frac{\partial u}{\partial t}(y_1, t) - c\frac{\partial^2 u}{\partial y_1^2}(y_1, t) - a\frac{\partial^2}{\partial y_1^2}\int_0^t u(y_1, s)ds + b\int_0^t u(y_1, s)ds \\ &= -\frac{\kappa}{E}\left(\lambda u(y_1, t) + (1-\lambda)\frac{\partial h}{\partial t}(y_1, t) - u_2(y_1, 1, t)\right)\end{aligned} \tag{3.5}$$

for any $0 < y_1 < L,\ 0 < t < T$, with the following boundary and initial conditions:

$$u_1(y_1, 1, t) = 0, \tag{3.6}$$

$$\begin{aligned}\left[\frac{\nu}{h}\frac{\partial u_2}{\partial y_2} - \nu\frac{\partial h}{\partial y_1}\left(\frac{\partial u_2}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_2}{\partial y_2}\right) - q\right](y_1, 1, t) = \\ \left(-q_w + \frac{1}{2}u_2\left(u_2 - \frac{\partial h}{\partial t}\right) - \kappa\left(u_2 - \lambda u - (1-\lambda)\frac{\partial h}{\partial t}\right)\right)(y_1, 1, t) \\ \left[\frac{1}{h}\frac{\partial q}{\partial y_2} - \frac{\partial h}{\partial y_1}\left(\frac{\partial q}{\partial y_1} - \frac{1}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right](y_1, 1, t) = \\ -\frac{1}{2}\frac{\partial h}{\partial t}(y_1, t)\ q(y_1, 1, t)\end{aligned} \tag{3.7}$$

for any $0 < y_1 < L,\ 0 < t < T$,

$$\nu\left(\frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2} - q\right)(L, y_2, t) = \left(-q_{out} + \frac{1}{2}|\boldsymbol{u}|^2\right)(L, y_2, t),$$
$$u_2(L, y_2, t) = 0, \qquad \left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)(L, y_2, t) = 0 \tag{3.8}$$

for any $0 < y_2 < 1$, $0 < t < T$,

$$\nu\left(\frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2} - q\right)(0, y_2, t) = \left(-q_{in} + \frac{1}{2}|\boldsymbol{u}|^2\right)(0, y_2, t)$$
$$u_2(0, y_2, t) = 0, \qquad \left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)(0, y_2, t) = 0 \tag{3.9}$$

for any $0 < y_2 < 1$, $0 < t < T$,

$$\nu\frac{\partial u_1}{\partial y_2}(y_1, 0, t) = 0, \qquad u_2(y_1, 0, t) = 0, \qquad \varepsilon\frac{\partial q}{\partial y_2}(y_1, 0, t) = 0$$

for any $0 < y_1 < L$, $0 < t < T$,

$$\boldsymbol{u}(y_1, y_2, 0) = \boldsymbol{0}, \qquad q(y_1, y_2, 0) = 0 \tag{3.10}$$

for any $0 < y_1 < L$, $0 < y_2 < 1$,

$$u(y_1, 0) = 0 \tag{3.11}$$

for any $0 < y_1 < L$ and finally,

$$u(0, t) = u(L, t) = 0 \tag{3.12}$$

for any $0 < t < T$.

3.2 Weak formulation of auxiliary problem

We continue this section by clarifying the meaning of the solution to the problem (3.1)–(3.12). We first define the space

$$V \equiv \boldsymbol{V} \times H^1(D) \times H_0^1(0, L) \tag{3.13}$$

where

$$\begin{aligned}
\boldsymbol{V} &\equiv \left\{\boldsymbol{w} \in H^1(D)^2 :\ w_1 = 0 \text{ on } S_w \text{ and } w_2 = 0 \text{ on } S_{in} \cup S_{out} \cup S_c\right\},\\
S_w &= \{(y_1, 1) : 0 < y_1 < L\},\\
S_{in} &= \{(0, y_2) : 0 < y_2 < 1\},\\
S_{out} &= \{(L, y_2) : 0 < y_2 < 1\},\\
S_c &= \{(y_1, 0) : 0 < y_1 < L\}.
\end{aligned} \tag{3.14}$$

For given functions of the boundary pressure and the domain deformation we assume

$$\begin{aligned}
&0 < \alpha \leq h(x_1,t) \leq \alpha^{-1}, \\
&h \in W^{2,2}((0,L)\times(0,T)) \cap C^1([0,L]\times[0,T]), \\
&q_{in},\ q_{out}\ \in L^2(0,T;L^2(0,1)), \\
&q_w\ \in L^2(0,T;L^2(0,L)).
\end{aligned} \tag{3.15}$$

In the sequel, if necessary we shall consider any function defined almost everywhere on $D\times(0,T)$ to be extended outside of $D\times(0,T)$ by zero.

We recall the notation (3.4), i.e.

$$\operatorname{div}_h \boldsymbol{u} \overset{\text{def}}{=} \frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2} + \frac{1}{h}\frac{\partial u_2}{\partial y_2}.$$

Definition 3.1 (Weak solution)**.** *We call $(\boldsymbol{u},q,u) \in L^2(0,T;V)$ a weak solution to the initial boundary value problem (3.1)–(3.12) if the following two properties are fulfilled:*

1. $\boldsymbol{u} \in L^\infty(0,T;L^2(D)^2)$, $\frac{\partial(h\boldsymbol{u})}{\partial t} \in \left(L^2(0,T;\boldsymbol{V}) \cap L^4(0,T;L^4(D)^2)\right)^*$, i.e. $\frac{\partial(h\boldsymbol{u})}{\partial t} \in (L^2(0,T;\boldsymbol{V}^*) + L^{4/3}(0,T;L^{4/3}(D)^2))$, such that

$$\int_0^T \left\langle \frac{\partial(h\boldsymbol{u})}{\partial t}, \boldsymbol{\zeta} \right\rangle dt + \int_0^T \int_D h\boldsymbol{u}\cdot\frac{\partial\boldsymbol{\zeta}}{\partial t}\,dt = 0 \tag{3.16}$$

for every test function $\boldsymbol{\zeta} \in L^2(0,T;\boldsymbol{V}) \cap L^4(0,T;L^4(D)^2) \cap H^{1,1}(0,T;L^2(D)^2)$ with $\boldsymbol{\zeta}(T)=0$, $q \in L^\infty(0,T;L^2(D))$, $\dfrac{\partial(hq)}{\partial t} \in L^2(0,T;H^{-1}(D))$ and $u \in L^\infty(0,T;H_0^1(0,L)) \cap H^1(0,T;L^2(0,L))$.

2. $(\boldsymbol{u}, q, u)$ *satisfies the following equation:*

$$\begin{aligned}
&-\int_0^T \left\langle \frac{\partial (h\boldsymbol{u})}{\partial t}, \boldsymbol{\psi} \right\rangle dt \ = \\
&\int_0^T \int_D \Bigg(-\frac{\partial h}{\partial t} \frac{\partial (y_2 \boldsymbol{u})}{\partial y_2} \cdot \boldsymbol{\psi} + \left(h u_1 \left(\frac{\partial \boldsymbol{u}}{\partial y_1} - \frac{y_2}{h} \frac{\partial h}{\partial y_1} \frac{\partial \boldsymbol{u}}{\partial y_2} \right) + u_2 \frac{\partial \boldsymbol{u}}{\partial y_2} \right) \cdot \boldsymbol{\psi} \\
&\qquad + \frac{h}{2}\, \boldsymbol{u} \cdot \boldsymbol{\psi}\ div_h \boldsymbol{u} + \frac{\partial \boldsymbol{\psi}}{\partial y_1} \cdot \left[\nu h \left(\frac{\partial \boldsymbol{u}}{\partial y_1} - \frac{y_2}{h} \frac{\partial h}{\partial y_1} \frac{\partial \boldsymbol{u}}{\partial y_2} \right) \right] \\
&\qquad + \frac{\partial \boldsymbol{\psi}}{\partial y_2} \cdot \left[\frac{\nu}{h} \frac{\partial \boldsymbol{u}}{\partial y_2} - \nu y_2 \frac{\partial h}{\partial y_1} \left(\frac{\partial \boldsymbol{u}}{\partial y_1} - \frac{y_2}{h} \frac{\partial h}{\partial y_1} \frac{\partial \boldsymbol{u}}{\partial y_2} \right) \right] - h\, q\ div_h \boldsymbol{\psi} \Bigg)\, dy\, dt \\
&+ \int_0^T \int_0^1 h(L,t) \left(q_{out} - \frac{1}{2} |u_1|^2 \right) \psi_1\ (L, y_2, t)\, dy_2 dt \\
&- \int_0^T \int_0^1 h(0,t) \left(q_{in} - \frac{1}{2} |u_1|^2 \right) \psi_1\ (0, y_2, t)\, dy_2 dt \\
&+ \int_0^T \int_0^L \Bigg(q_w - \frac{1}{2} u_2 \left(u_2 - \frac{\partial h}{\partial t} \right) \\
&\qquad + \kappa \left(u_2 - \lambda u - (1-\lambda) \frac{\partial h}{\partial t} \right) \Bigg) \psi_2\ (y_1, 1, t)\, dy_1 dt \\
&+ \varepsilon \int_0^T \left\langle \frac{\partial (hq)}{\partial t}, \phi \right\rangle dt \qquad\qquad (3.17) \\
&+ \int_0^T \int_D \Bigg(-\varepsilon \frac{\partial h}{\partial t} \frac{\partial (y_2 q)}{\partial y_2} \phi + \varepsilon \frac{\partial \phi}{\partial y_1} \left[h \left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h} \frac{\partial h}{\partial y_1} \frac{\partial q}{\partial y_2} \right) \right] \\
&\qquad + \varepsilon \frac{\partial \phi}{\partial y_2} \left[\frac{1}{h} \frac{\partial q}{\partial y_2} - y_2 \frac{\partial h}{\partial y_1} \left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h} \frac{\partial h}{\partial y_1} \frac{\partial q}{\partial y_2} \right) \right] + \ h\ div_h \boldsymbol{u}\ \phi \Bigg)\, dy\, dt \\
&+ \frac{\varepsilon}{2} \int_0^T \int_0^L \frac{\partial h}{\partial t}(y_1, t)\ q\phi\ (y_1, 1, t)\, dy_1 dt \\
&+ \int_0^T \int_0^L \Bigg(\frac{\partial u}{\partial t} \xi + c \frac{\partial u}{\partial y_1} \frac{\partial \xi}{\partial y_1} + a \frac{\partial}{\partial y_1} \int_0^t u(y_1, s) ds \frac{\partial \xi}{\partial y_1} \\
&\qquad + \ b \int_0^t u(y_1, s) ds\ \xi + \frac{\kappa}{E} \left(\lambda u + (1-\lambda) \frac{\partial h}{\partial t} - u_2 \right) \xi \Bigg) (y_1, t)\, dy_1 dt
\end{aligned}$$

for every $(\boldsymbol{\psi}, \phi, \xi) \in L^2(0,T;V)$, $\boldsymbol{\psi} \in L^4(0,T;L^4(D)^2)$.

Note that (3.16) implies

$$\int_0^\tau \left\langle \frac{\partial (h\boldsymbol{u})}{\partial t}, \boldsymbol{\zeta} \right\rangle dt + \int_0^\tau \int_D h\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\zeta}}{\partial t}\, dy\, dt = \int_D h\boldsymbol{u} \cdot \boldsymbol{\zeta}\ (\tau)\, dy \qquad (3.18)$$

and that (3.17) holds for almost all $\tau \in (0,T)$ if T is replaced by τ. To prove this, replace $\boldsymbol{\zeta}$ in (3.16) with $\boldsymbol{\zeta}(y,t)\chi_\epsilon(t)$, where

$$\chi_\epsilon(t) = \max\left\{0, \min\left\{1, \frac{\tau+\epsilon-t}{\epsilon}\right\}\right\}$$

Note that this is an admissible test function. Then by passing to the limit, as $\epsilon \to 0$ in (3.16) we obtain (3.18) for almost all τ.

We will use the following form of the interpolation theorems, which play an important role by proving the existence of the weak solution:

Proposition 3.1. *Let φ be any function in $H^1(D)$ such that $\varphi = 0$ on S_w or $\varphi = 0$ on S_s. Then for any $p \geq 2$ and for any number θ in the interval*

$$\frac{p-2}{p} \leq \theta \leq 1$$

a constant $C = C(p,\theta)$ exists such that

$$\|\varphi\|_{L^p(D)} \leq C \|\nabla\varphi\|^{\theta}_{L^2(D)} \|\varphi\|^{1-\theta}_{L^2(D)}. \tag{3.19}$$

Moreover, if φ be any function in $L^2(0,T;H^1(D)) \cap L^\infty(0,T;L^2(D))$ such that $\varphi = 0$ on S_w or $\varphi = 0$ on S_s for almost all t, then for any $p \geq 2$

$$\|\varphi\|_{L^{\frac{2p}{p-2}}(0,T;L^p(D))} \leq C\left[\|\varphi\|_{L^\infty(0,T;L^2(D))}\right]^{\frac{2}{p}} \left[\|\varphi\|_{L^2(0,T;H^1(D))}\right]^{\frac{p-2}{p}}. \tag{3.20}$$

Proof of Proposition 3.1. The form of Nierenberg–Gagliardo inequality (3.19) can be found e.g. in [Hen81]. Then (3.20) follows from (3.19) for $\theta = \frac{p-2}{p}$ by integration over $(0,T)$. ∎

Proposition 3.2. *Denote $S \equiv S_{in} \cup S_s \cup S_{out} \cup S_w$ and let φ be any function in $H^1(D)$ such that $\varphi = 0$ on S_w or $\varphi = 0$ on S_s. Then for any $r \geq 2$ a constant $C = C(r)$ exists such that*

$$\|\varphi\|_{L^r(S)} \leq C \|\nabla\varphi\|^{1-\frac{1}{r}}_{L^2(D)} \|\varphi\|^{\frac{1}{r}}_{L^2(D)}. \tag{3.21}$$

Moreover, if φ be any function in $L^2(0,T;H^1(D)) \cap L^\infty(0,T;L^2(D))$ such that $\varphi = 0$ on S_w or $\varphi = 0$ on S_s for almost all t then for any $r \geq 2$

$$\|\varphi\|_{L^{\frac{2r}{r-1}}(0,T;L^r(S))} \leq C\left[\|\varphi\|_{L^\infty(0,T;L^2(D))}\right]^{\frac{1}{r}} \left[\|\varphi\|_{L^2(0,T;H^1(D))}\right]^{\frac{r-1}{r}}. \tag{3.22}$$

Proof of Proposition 3.2. Let $\varphi = 0$ on S_s and $\varphi \in C^1(\overline{D})$. For $0 < y_2 < 1$ and $0 < y_1 < L$, one easily gets

$$|\varphi(0, y_2)|^r = -\int_0^{y_1} \frac{\partial}{\partial \sigma} \left(|\varphi(\sigma, y_2)|^r\right) \, d\sigma + |\varphi(y_1, y_2)|^r.$$

Thus,

$$|\varphi(0, y_2)|^r \leq r \int_0^L |\varphi(\sigma, y_2)|^{r-1} \left| \frac{\partial \varphi}{\partial y_1}(\sigma, y_2) \right| \, d\sigma + |\varphi(y_1, y_2)|^r$$

and integration over $(0, 1)$ through y_2 and over $(0, L)$ through y_1 yields

$$\|\varphi\|_{L^r(S_{in})} \leq (r)^{1/r} \, \|\nabla \varphi\|_{L^2(D)}^{\frac{1}{r}} \, \|\varphi\|_{L^{2(r-1)}(D)}^{\frac{r-1}{r}} + L^{-1/r} \, \|\varphi\|_{L^r(D)} \, . \tag{3.23}$$

Now we apply the Nierenberg-Gagliardo inequality (3.19) for $p = r$ and for $p = 2(r-1)$. (3.23) then yields

$$\|\varphi\|_{L^r(S_{in})} \leq C_1 \, \|\nabla \varphi\|_{L^2(D)}^{\frac{r-1}{r}} \, \|\varphi\|_{L^2(D)}^{\frac{1}{r}} + C_2 \, \|\nabla \varphi\|_{L^2(D)}^{\frac{r-2}{r}} \, \|\varphi\|_{L^2(D)}^{\frac{2}{r}} \, .$$

Finally, (3.21) follows from an estimation of $\|\varphi\|_{L^2(D)}^{\frac{1}{r}}$ using the Nierenberg-Gagliardo inequality (3.19) for $p = 2$ and $\theta = 1$.
(3.22) follows, similarly as above, from (3.21) by integration over $(0, T)$. ■

Chapter 4

Existence and uniqueness

We now turn to the existence and uniqueness of a weak solution defined in Definition 3.1. The proof of existence is carried out in several steps with the use of Rothe's method, based on semidiscretisation in time (Rothe's method is extensively studied e.g. in [Kač85]). The existence and uniqueness of the Navier-Stokes problem is also studied in [Fei93], or [Tem79]. However, these works deal neither with the Neumann-type boundary condition, nor with the perturbation of the Navier-Stokes system coupled with time-dependent domain deformation.

Our approach differs in technical details from the approach of Feistauer [Fei93]. For example, we can not assume a divergence-free velocity field, because the divergence operator depends on the domain deformation function $h(x_1, t)$ and consequently on time (in [Fei93], a divergence-free velocity field is assumed). Some difficulties appear by proving the time compactness of the solution, see Remark 4.2, Remark 4.4 and also by proving the uniqueness, which is introduced later in this chapter. The regularisation of the divergence-free condition helps us to overcome these difficulties.

The time discretisation introduced in Section 4.1 converts the non-stationary problem to a sequence of stationary problems and allows us to construct a sequence of approximate functions. We first prove the existence of the weak solution for the stationary problem in Subsection 4.1.1. In Section 4.3, we prove the convergence of approximate piecewise linear and piecewise constant functions to the solution in corresponding function spaces with the assistance of a priori estimates from Section 4.2. This proof is based on the compactness of the solution in the corresponding function spaces. The compactness is implied by L^1 equicontinuity. We give a proof of equicontinuity in Theorem 4.3, where we follow the idea of [AL83]. (A different technique of proving the compactness is taken in [Tem79].) Then we prove the existence of the distributive derivate $\frac{\partial(h\boldsymbol{u})}{\partial t} \in X^* = L^2(0,T;\boldsymbol{V}^*) + L^{4/3}(0,T;L^{4/3}(D)^2)$ similarly as Temam [Tem79]. In Section 4.4, we prove uniqueness of the solution. This uniqueness is continuously dependent on the data such as domain deformation h and the boundary pressure. We conclude this chapter with Section 4.5, where we prove the existence of the weak solution $(\boldsymbol{u}, u)$ for problem (1.1)–(1.10), i.e. the problem with divergence-free velocities, it means we let $\varepsilon \longrightarrow 0$.

4.1 Time discretisation

To prove the existence of the solution to (3.1)–(3.12), we approximate the problem by a sequence of stationary perturbed Navier-Stokes systems coupled with the parabolic problem for pressure and deformation of the wall. This approach is important also for the numerical analysis of the problem. We replace

$$\frac{\partial(hu_k)}{\partial t}, \frac{\partial(hq)}{\partial t} \text{ and } \frac{\partial u}{\partial t}$$

with backward difference quotients

$$\frac{h^i u_k^i - h^{i-1}u_k^{i-1}}{\Delta t}, \quad \frac{h^i q^i - h^{i-1}q^{i-1}}{\Delta t} \quad \text{and} \quad \frac{u^i - u^{i-1}}{\Delta t}$$

for $\Delta t \equiv T/n$, $n \in N$, $n \gg 1$ and we replace

$$\int_0^t u(s)\, ds \quad \text{with} \quad \sum_{k=1}^{i} u^k \Delta t$$

for $i\Delta t \leq t < (i+1)\Delta t$, where u_k^i, q^i and u^i denote approximations of unknown u_k, q and u at time $i\Delta t$.

In this chapter, we use the following notation $h^i(y_1) = h(y_1, i\Delta t)$,

$$\operatorname{div}_i \boldsymbol{u} \stackrel{\text{def}}{=} \frac{\partial u_1}{\partial y_1} - \frac{y_2}{h^i}\frac{\partial h^i}{\partial y_1}\frac{\partial u_1}{\partial y_2} + \frac{1}{h^i}\frac{\partial u_2}{\partial y_2},$$

$$q_{in}^i(y_2) = \frac{1}{\Delta t}\int_{(i-1)\Delta t}^{i\Delta t} q_{in}(y_2, s)\, ds, \qquad q_{out}^i(y_2) = \frac{1}{\Delta t}\int_{(i-1)\Delta t}^{i\Delta t} q_{out}(y_2, s)\, ds$$

and

$$q_w^i(y_1) = \frac{1}{\Delta t}\int_{(i-1)\Delta t}^{i\Delta t} q_w(y_1, s)\, ds.$$

Hence, for each $i \in \{1, 2, \ldots, n\}$ and given $(\boldsymbol{u}^j, q^j, u^j)$, $0 \leq j \leq i-1$ we get a perturbed stationary Navier-Stokes system for $\boldsymbol{u}^i$ coupled with parabolic problems for q^i and u^i, $i = 1, 2, \ldots, n$, which in the variational formulation reads as follows:

$$
\begin{aligned}
&\int_D \Bigg\{ \left(\frac{h^i \boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}}{\Delta t} - \frac{h^i - h^{i-1}}{\Delta t} \frac{\partial (y_2 \boldsymbol{u}^i)}{\partial y_2} \right) \cdot \boldsymbol{\omega} \\
&\quad + \left(h^i u_1^i \left(\frac{\partial \boldsymbol{u}^i}{\partial y_1} - \frac{y_2}{h^i} \frac{\partial h^i}{\partial y_1} \frac{\partial \boldsymbol{u}^i}{\partial y_2} \right) + u_2^i \frac{\partial \boldsymbol{u}^i}{\partial y_2} \right) \cdot \boldsymbol{\omega} + \frac{h^i}{2} \boldsymbol{u}^i \cdot \boldsymbol{\omega} \, \mathrm{div}_i \, \boldsymbol{u}^i \\
&\quad + \nu \frac{\partial \boldsymbol{\omega}}{\partial y_1} \cdot \left[h^i \left(\frac{\partial \boldsymbol{u}^i}{\partial y_1} - \frac{y_2}{h^i} \frac{\partial h^i}{\partial y_1} \frac{\partial \boldsymbol{u}^i}{\partial y_2} \right) \right] \\
&\quad + \nu \frac{\partial \boldsymbol{\omega}}{\partial y_2} \cdot \left[\frac{1}{h^i} \frac{\partial \boldsymbol{u}^i}{\partial y_2} - y_2 \frac{\partial h^i}{\partial y_1} \left(\frac{\partial \boldsymbol{u}^i}{\partial y_1} - \frac{y_2}{h^i} \frac{\partial h^i}{\partial y_1} \frac{\partial \boldsymbol{u}^i}{\partial y_2} \right) \right] - h^i q^i \, \mathrm{div}_i \, \boldsymbol{\omega} \Bigg\} \, dy \\
&+ \int_D \Bigg\{ \varepsilon \left(\frac{h^i q^i - h^{i-1} q^{i-1}}{\Delta t} - \frac{h^i - h^{i-1}}{\Delta t} \frac{\partial (y_2 q^i)}{\partial y_2} \right) v \\
&\quad + \varepsilon \frac{\partial v}{\partial y_1} \left[h^i \left(\frac{\partial q^i}{\partial y_1} - \frac{y_2}{h^i} \frac{\partial h^i}{\partial y_1} \frac{\partial q^i}{\partial y_2} \right) \right] \\
&\quad + \varepsilon \frac{\partial v}{\partial y_2} \left[\frac{1}{h^i} \frac{\partial q^i}{\partial y_2} - y_2 \frac{\partial h^i}{\partial y_1} \left(\frac{\partial q^i}{\partial y_1} - \frac{y_2}{h^i} \frac{\partial h^i}{\partial y_1} \frac{\partial q^i}{\partial y_2} \right) \right] + h^i \, v \, \mathrm{div}_i \, \boldsymbol{u}^i \Bigg\} \, dy \\
&+ \int_0^1 \ell \left(q^i_{out} - \frac{1}{2} \left| u_1^i \right|^2 \right) \omega_1 \; (L, y_2) \, dy_2 \\
&- \int_0^1 \ell \left(q^i_{in} - \frac{1}{2} \left| u_1^i \right|^2 \right) \omega_1 \; (0, y_2) \, dy_2 \\
&+ \int_0^L \left(q^i_w - \frac{1}{2} u_2^i \left(u_2^i - \frac{h^i - h^{i-1}}{\Delta t} \right) \right) \omega_2 \; (y_1, 1) \, dy_1 \\
&+ \int_0^L \left(\kappa \left(u_2^i - \lambda u^i - (1-\lambda) \frac{h^i - h^{i-1}}{\Delta t} \right) \omega_2 + \frac{\varepsilon}{2} \frac{h^i - h^{i-1}}{\Delta t} q^i v \right) (y_1, 1) \, dy_1 \\
&+ \int_0^L \Bigg(\frac{u^i - u^{i-1}}{\Delta t} \vartheta + c \frac{\partial u^i}{\partial y_1} \frac{\partial \vartheta}{\partial y_1} + a \sum_{k=1}^{i} \frac{\partial u^k}{\partial y_1} \Delta t \frac{\partial \vartheta}{\partial y_1} \\
&\quad + b \left(\sum_{k=1}^{i} u^k \Delta t \right) \vartheta + \frac{\kappa}{E} \left(\lambda u^i + (1-\lambda) \frac{h^i - h^{i-1}}{\Delta t} - u_2^i \right) \vartheta \Bigg) (y_1) \, dy_1 \;\; = \;\; 0
\end{aligned}
\tag{4.1}
$$

for any $\boldsymbol{\varpi} = (\boldsymbol{\omega}, v, \vartheta) \in V$.

4.1.1 Existence of the solution for the stationary problem

In this section we study the following variational problem:

$$
\text{Find } \boldsymbol{w}^i = (\boldsymbol{u}^i, q^i, u^i) \in V \text{ such that}
$$

$$
a^i(\boldsymbol{w}^i, \boldsymbol{\varpi}) + b^i(\boldsymbol{w}^i, \boldsymbol{w}^i, \boldsymbol{\varpi}) = L^i(\boldsymbol{\varpi}) \qquad \forall \boldsymbol{\varpi} \in V, \tag{4.2}
$$

where $\boldsymbol{\varpi} = (\boldsymbol{\omega}, v, \vartheta)$, V is defined by (3.13) and $a^i(\cdot,\cdot)$, $b^i(\cdot,\cdot,\cdot)$, $L^i(\cdot)$ are determined by (4.1), i.e.

1. $a^i(\cdot,\cdot) : V \times V \to R$ is the following bilinear continuous form on V:

$$\begin{aligned} a^i(\boldsymbol{w}^i, \boldsymbol{\varpi}) \;=\; & \nu a_1^i(u_1^i, \omega_1) + \nu a_1^i(u_2^i, \omega_2) + \frac{1}{\Delta t}\int_D h^i\, \boldsymbol{u}^i \cdot \boldsymbol{\omega}\, dy \\ & + \varepsilon a_1^i(q^i, v) + \frac{\varepsilon}{\Delta t}\int_D h^i\, q^i\, v\, dy \\ & + \int_0^L \left((c + a\Delta t)\frac{\partial u^i}{\partial y_1}\frac{\partial \vartheta}{\partial y_1} + \left(\frac{1}{\Delta t} + b\Delta t\right) u^i \vartheta \right) dy_1 \\ & - \int_D \frac{h^i - h^{i-1}}{\Delta t}\, \frac{\partial (y_2 \boldsymbol{u}^i)}{\partial y_2} \cdot \boldsymbol{\omega}\, dy + \int_0^L \frac{1}{2} u_2^i \frac{h^i - h^{i-1}}{\Delta t} \omega_2\, (y_1, 1)\, dy_1 \\ & - \varepsilon \int_D \frac{h^i - h^{i-1}}{\Delta t}\, \frac{\partial (y_2 q^i)}{\partial y_2} v\, dy + \frac{\varepsilon}{2}\int_0^L \frac{h^i - h^{i-1}}{\Delta t}\, q^i\, v\, (y_1, 1)\, dy_1 \\ & + \kappa \int_0^L (\lambda u^i - u_2^i)\left(\frac{\vartheta}{E} - \omega_2\right)(y_1)\, dy_1 \\ & + \int_D \left(h^i\, v \operatorname{div}_i \boldsymbol{u}^i - h^i q^i \operatorname{div}_i \boldsymbol{\omega} \right) dy \end{aligned}$$

where

$$\begin{aligned} a_1^i(q, v) = \int_D \Bigg\{ & \left[h^i \left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h^i}\frac{\partial h^i}{\partial y_1}\frac{\partial q}{\partial y_2} \right) \right] \frac{\partial v}{\partial y_1} \\ & + \left[\frac{1}{h^i}\frac{\partial q}{\partial y_2} - y_2 \frac{\partial h^i}{\partial y_1}\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h^i}\frac{\partial h^i}{\partial y_1}\frac{\partial q}{\partial y_2} \right) \right] \frac{\partial v}{\partial y_2} \Bigg\}\, dy. \end{aligned} \tag{4.3}$$

2. The trilinear form $b^i(\cdot,\cdot,\cdot)$ is defined by

$$b^i(\cdot,\cdot,\cdot) : \; V \times V \times V \longrightarrow R$$

$$\begin{aligned} b^i(\boldsymbol{w}^i, \boldsymbol{m}^i, \boldsymbol{\varpi}) \;=\; & \int_D \Bigg\{ \left(h^i u_1^i \left(\frac{\partial \boldsymbol{z}^i}{\partial y_1} - \frac{y_2}{h^i}\frac{\partial h^i}{\partial y_1}\frac{\partial \boldsymbol{z}^i}{\partial y_2} \right) + u_2^i \frac{\partial \boldsymbol{z}^i}{\partial y_2} \right) \cdot \boldsymbol{\omega} \\ & + \frac{h^i}{2}\, \boldsymbol{z}^i \cdot \boldsymbol{\omega} \operatorname{div}_i \boldsymbol{u}^i \Bigg\}\, dy \\ & - \frac{1}{2}\int_0^1 \ell u_1 z_1 \omega_1\, (L, y_2)\, dy_2 + \frac{1}{2}\int_0^1 \ell u_1 z_1 \omega_1\, (0, y_2)\, dy_2 \\ & - \frac{1}{2}\int_0^L u_2 z_2 \omega_2\, (y_1, 1)\, dy_1 \end{aligned} \tag{4.4}$$

for $\boldsymbol{m}^i = (\boldsymbol{z}^i, \cdot, \cdot)$, $\boldsymbol{w}^i = (\boldsymbol{u}^i, \cdot, \cdot)$, $\boldsymbol{\varpi} = (\boldsymbol{\omega}, \cdot, \cdot)$.

3. Finally, $L^i(\cdot)$ is the linear functional on V,

$$\begin{aligned} L^i(\boldsymbol{\varpi}) &= \frac{1}{\Delta t}\int_D h^{i-1}\left(\boldsymbol{u}^{i-1}\cdot\boldsymbol{\omega} + \varepsilon\, q^{i-1}\, v\right)\, dy + \frac{1}{\Delta t}\int_0^L u^{i-1}\vartheta\, dy_1 \\ &\quad + \int_0^1 \ell\left(q^i_{in}\,\omega_1\,(0,y_2) - q^i_{out}\,\omega_1\,(L,y_2)\right)\, dy_2 \\ &\quad + \int_0^L \left(-q^i_w\,\omega_2\,(y_1,1) - \sum_{k=1}^{i-1}\left(a\frac{\partial u^k}{\partial y_1}\frac{\partial \vartheta}{\partial y_1} + bu^k\vartheta\right)(y_1)\Delta t\right)\, dy_1 \\ &\quad + \kappa(1-\lambda)\int_0^L \frac{h^i - h^{i-1}}{\Delta t}\left(\omega_2 - \frac{\vartheta}{E}\right)(y_1)dy_1. \end{aligned}$$

We now prove the existence of the weak solution for variational problem (4.2).

Theorem 4.1 (Stationary solution)**.** *Let $i \in \{1,2,\ldots,n\}$ and $\boldsymbol{w}^j \in V$ for $j \le i-1$ be given. Assume there are non-negative constants α, K, independent on i, such that*

$$0 < \alpha \le h^i(y_1) \le \alpha^{-1}, \ \textit{see (3.15)} \tag{4.5}$$

and

$$\left|\frac{\partial h^i}{\partial y_1}(y_1)\right| + \left|\frac{h^i(y_1) - h^{i-1}(y_1)}{\Delta t}\right| \le K \tag{4.6}$$

for all $0 \le y_1 \le L$ and $i = 1,2,\ldots n$. Moreover, assume that

$$q^i_{in}, q^i_{out} \in L^2(0,1),\ q^i_w \in L^2(0,L) \quad \textit{and} \quad \Delta t \le \alpha/K.$$

Then the problem (4.2) has at least one solution.

Proof of Theorem 4.1. Following [CMP94, Proof of Theorem 2.1], we use Galerkin's method. V is a closed subspace of $H^1(D)^3 \times H^1_0(0,L)$ and it is thus possible to choose a basis $\{\boldsymbol{\zeta}_k\}_{k=1}^{\infty} \subset V$. For every $\ell \in N$, $\ell \ge 1$ we define an approximate problem as:

$$\text{Find } c_{k\ell} \in R,\ 1 \le k \le \ell \text{ such that } \boldsymbol{w}_\ell = \sum_{k=1}^{\ell} c_{k\ell}\,\boldsymbol{\zeta}_k \text{ is a solution of}$$

$$a^i(\boldsymbol{w}_\ell, \boldsymbol{\zeta}_k) + b^i(\boldsymbol{w}_\ell, \boldsymbol{w}_\ell, \boldsymbol{\zeta}_k) = L^i(\boldsymbol{\zeta}_k) \qquad \forall k = 1,\ldots,\ell. \tag{4.7}$$

To prove the existence of a solution to (4.7), we use the following lemma (see [Lio69, Lemma 1.4.3, p. 53] or [Tem79, Lemma 2.1.4, p. 164]).

Lemma 4.1. *Let Y be a finite-dimensional Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let P be a continuous mapping from Y into itself, such that for a sufficiently large $\rho > 0$,*

$$(P(\zeta),\zeta) \ge 0 \quad \forall \zeta \in Y \ \textit{such that}\ \|\zeta\| = \rho. \tag{4.8}$$

Then $\zeta \in Y$, $\|\zeta\| \le \rho$ exists such that $P(\zeta) = 0$.

In our case $Y = Y_\ell = \text{span}\{\boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_\ell\}$ equipped with the scalar product of $H^1(D)^3 \times H_0^1(0,L)$, and for any $\boldsymbol{\zeta} \in Y$, $P(\boldsymbol{\zeta}) = P_\ell(\boldsymbol{\zeta}) \in Y$ is defined using Rietz's theorem as

$$(P_\ell(\boldsymbol{\zeta}), \boldsymbol{z}) = a^i(\boldsymbol{\zeta}, \boldsymbol{z}) + b^i(\boldsymbol{\zeta}, \boldsymbol{\zeta}, \boldsymbol{z}) - L^i(\boldsymbol{z}) \quad \forall \boldsymbol{z} \in Y_\ell.$$

In order to understand why P_ℓ is continuous, we subtract the previous identities for $\boldsymbol{\zeta}^1$, $\boldsymbol{\zeta}^2 \in Y_\ell$, then we estimate right-hand side using the relation (4.12) below and we obtain

$$\begin{aligned}(P_\ell(\boldsymbol{\zeta}^1) - P_\ell(\boldsymbol{\zeta}^2), \boldsymbol{z}) &= a^i(\boldsymbol{\zeta}^1 - \boldsymbol{\zeta}^2, \boldsymbol{z}) + b^i(\boldsymbol{\zeta}^1 - \boldsymbol{\zeta}^2, \boldsymbol{\zeta}^1, \boldsymbol{z}) + b^i(\boldsymbol{\zeta}^2, \boldsymbol{\zeta}^1 - \boldsymbol{\zeta}^2, \boldsymbol{z}) \\ &\le c_1 \|z\|_V \|\boldsymbol{\zeta}^1 - \boldsymbol{\zeta}^2\|_V + c_2 \|\boldsymbol{\zeta}^1 - \boldsymbol{\zeta}^2\|_{L^4(D)} \|\nabla \boldsymbol{\zeta}^1\|_{L^2(D)} \|z\|_{L^4(D)} \\ &\quad + c_3 \|\boldsymbol{\zeta}^2\|_{L^4(D)} \|\nabla \boldsymbol{\zeta}^1 - \nabla \boldsymbol{\zeta}^2\|_{L^2(D)} \|z\|_{L^4(D)},\end{aligned}$$

where the norms $\|.\|_{L^q(D)} = \|.\|_{L^q(D)^4}$. By putting $z = P_\ell(\boldsymbol{\zeta}^1) - P_\ell(\boldsymbol{\zeta}^2)$ and since V is embedded into $L^q(D)^4$ for $\forall q \ge 2$ (see Proposition 3.1), we obtain that $\|P_\ell(\boldsymbol{\zeta}^1) - P_\ell(\boldsymbol{\zeta}^2)\|_V \to 0$ as $\|\boldsymbol{\zeta}^1 - \boldsymbol{\zeta}^2\|_V \to 0$.

In order to prove (4.8), we first introduce the following lemma.

Lemma 4.2. *Let (4.5)–(4.6) be satisfied. Then*

$$a_1^i(v,v) \ge \frac{\alpha}{2+K^2} \int_D |\nabla v|^2 \, dy. \tag{4.9}$$

for any $v \in H^1(D)^2$, where a_1^i is given by (4.3).

Proof of Lemma 4.2. First, note that

$$a_1^i(v,v) = \int_D \left\{ h^i \left(\frac{\partial v}{\partial y_1} - \frac{y_2}{h^i} \frac{\partial h^i}{\partial y_1} \frac{\partial v}{\partial y_2} \right)^2 + \frac{1}{h^i} \left(\frac{\partial v}{\partial y_2} \right)^2 \right\} dy. \tag{4.10}$$

For a moment, denote

$$A = \sqrt{h^i} \frac{\partial v}{\partial y_1}, \; B = \frac{1}{\sqrt{h^i}} \frac{\partial v}{\partial y_2}, \; z = y_2 \frac{\partial h}{\partial y_1}$$

and rewrite

$$a_1^i(v,v) = \int_D \left\{ A^2 - 2zAB + (1+z^2)B^2 \right\} dy.$$

Choosing $0 < \delta < 1/(K^2+1)$ and using Young's inequality $2zAB \le A^2(1-\delta) + B^2 \frac{z^2}{(1-\delta)}$ yields

$$A^2 - 2zAB + (1+z^2)B^2 \ge \delta A^2 + \frac{1-\delta(z^2+1)}{1-\delta} B^2.$$

As $|z| < K$, after putting $\delta = 1/(K^2+2)$, (4.9) follows easily. ∎

Next, note that

$$(P_\ell(\boldsymbol{\zeta}),\boldsymbol{\zeta}) = a^i(\boldsymbol{\zeta},\boldsymbol{\zeta}) - L^i(\boldsymbol{\zeta}),$$

as

$$\begin{aligned}
b^i(\boldsymbol{\zeta},\boldsymbol{\zeta},\boldsymbol{\zeta}) \quad &= \\
&\int_D \left[\left(h^i v_1\left(\frac{\partial \boldsymbol{v}}{\partial y_1} - \frac{y_2}{h^i}\frac{\partial h^i}{\partial y_1}\frac{\partial \boldsymbol{v}}{\partial y_2}\right) + v_2\frac{\partial \boldsymbol{v}}{\partial y_2}\right)\cdot\boldsymbol{v} + \frac{h^i}{2}\left|\boldsymbol{v}\right|^2 \mathrm{div}_i\boldsymbol{v}\right] dy \\
&- \int_0^1 hv_1\left|\boldsymbol{v}\right|^2(L,y_2)\,dy_2 + \int_0^1 hv_1\left|\boldsymbol{v}\right|^2(0,y_2)\,dy_2 \\
&- \int_0^L v_2\left|\boldsymbol{v}\right|^2(y_1,1)\,dy_1 \;=\; 0 \qquad (4.11)
\end{aligned}$$

for $\boldsymbol{\zeta} = (\boldsymbol{v}, p, \lambda Ev)$. In order to obtain (4.11), we used the fact that

$$b^i(\boldsymbol{w}^i,\boldsymbol{m}^i,\boldsymbol{\varpi}) = \frac{1}{2}B^i(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\omega}) - \frac{1}{2}B^i(\boldsymbol{u},\boldsymbol{\omega},\boldsymbol{z}) \qquad (4.12)$$

for $\boldsymbol{m}^i = (\boldsymbol{z}^i,\cdot,\cdot)$, $\boldsymbol{w}^i = (\boldsymbol{u}^i,\cdot,\cdot)$, $\boldsymbol{\varpi} = (\boldsymbol{\omega},\cdot,\cdot)$, where

$$B^j(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\omega}) \equiv \int_D \left(h^j u_1\left(\frac{\partial \boldsymbol{z}}{\partial y_1} - \frac{y_2}{h^j}\frac{\partial h^j}{\partial y_1}\frac{\partial \boldsymbol{z}}{\partial y_2}\right) + u_2\frac{\partial \boldsymbol{z}}{\partial y_2}\right)\cdot\boldsymbol{\omega}\,dy.$$

To obtain (4.12), per partes integration in part $\int_D \frac{h^i}{2}\boldsymbol{z}^i, \boldsymbol{\omega}\mathrm{div}_i\boldsymbol{u}^i\,dy$ of nonlinear term $b^i(\boldsymbol{w}^i,\boldsymbol{m}^i,\boldsymbol{\varpi})$ has to be performed.

Next, it is easy to see that a positive constant C exists such that

$$\left|L^i(\boldsymbol{\zeta})\right| \le C\|\boldsymbol{\zeta}\|_V \qquad \forall\boldsymbol{\zeta}\in V. \qquad (4.13)$$

Finally, one can verify that for $\boldsymbol{\zeta} = (\boldsymbol{v}, p, \lambda Ev)$

$$\begin{aligned}
a^i(\boldsymbol{\zeta},\boldsymbol{\zeta}) \quad &= \quad \nu a_1^i(v_1,v_1) + \nu a_1^i(v_2,v_2) + \varepsilon a_1^i(p,p) \qquad (4.14)\\
&+ \int_D \left[\frac{h^i}{\Delta t} + \frac{y_2}{2}\frac{h^i - h^{i-1}}{\Delta t}\right]\left(|\boldsymbol{v}|^2 + \varepsilon|p|^2\right)\,dy \\
&+ \int_0^L \left((c + a\Delta t)\left|\frac{\partial v}{\partial y_1}\right|^2 + \left(\frac{1}{\Delta t} + b\Delta t\right)|v|^2 + \kappa(v_2 - \lambda v)^2\right)dy_1
\end{aligned}$$

and therefore, if Δt is sufficiently small, e.g. $0 < \Delta t < \frac{\alpha}{K}$, then a constant $\delta > 0$ exists such that

$$a^i(\boldsymbol{\zeta},\boldsymbol{\zeta}) \ge \delta\|\boldsymbol{\zeta}\|_V^2 \qquad \forall\boldsymbol{\zeta}\in V. \qquad (4.15)$$

Hence

$$(P_\ell(\boldsymbol{\zeta}),\boldsymbol{\zeta}) \ge \delta\|\boldsymbol{\zeta}\|_V\left(\|\boldsymbol{\zeta}\|_V - \frac{C}{\delta}\right) \qquad \forall\boldsymbol{\zeta}\in Y_\ell$$

which implies that P_ℓ satisfies (4.8) with $\|\boldsymbol{\zeta}\|_V = \rho = C/\delta$. Thus, for any $\ell \in N$ a solution $\boldsymbol{w}_\ell$ of (4.7) exists which satisfies

$$\|\boldsymbol{w}_\ell\|_V \leq \rho.$$

The sequence $\{\boldsymbol{w}_\ell\}$ is bounded, therefore $\boldsymbol{w}^i \in V$ and a subsequence $\{\boldsymbol{w}_{\ell'}\}$ of $\{\boldsymbol{w}_\ell\}$ exist such that

$$\boldsymbol{w}_{\ell'} \to \boldsymbol{w}^i \text{ weakly in } V \text{ as } \ell' \to \infty \tag{4.16}$$

and due to the compact embedding of V into $\left(L^2(D)\right)^3 \times L^2(0,L)$,

$$\boldsymbol{w}_{\ell'} \to \boldsymbol{w}^i \text{ strongly in } \left(L^2(D)\right)^3 \times L^2(0,L) \text{ as } \ell' \to \infty. \tag{4.17}$$

From Proposition 3.1 and Proposition 3.2 then follows

$$\boldsymbol{w}_{\ell'} \to \boldsymbol{w}^i \text{ strongly in } (L^p(D))^3 \times L^p(0,L) \text{ as } \ell' \to \infty \tag{4.18}$$

and

$$\boldsymbol{w}_{\ell'} \to \boldsymbol{w}^i \text{ strongly in } (L^p(S))^3 \times L^p(0,L) \text{ as } \ell' \to \infty \tag{4.19}$$

for any $p \geq 2$. Since $\int_D u^q \leq \|u\|^q_{L^2(D)} |D|^{\frac{2-q}{2}}$ for $1 \leq q < 2$, we have (4.18), (4.19) also for $p \geq 1$. As for test functions, let $\boldsymbol{\varpi} \in V$ and a sequence $\{\boldsymbol{\varpi}_\ell\}$ be such that $\boldsymbol{\varpi}_\ell \in Y_\ell$ and

$$\boldsymbol{\varpi}_\ell \to \boldsymbol{\varpi} \text{ strongly in } V \tag{4.20}$$

as $\ell \to \infty$. Note that $\boldsymbol{\varpi}$ converges also in the spaces from (4.18) and (4.19).

Finally, according to (4.2), it holds for every ℓ' that

$$a^i(\boldsymbol{w}_{\ell'}, \boldsymbol{\varpi}_{\ell'}) + b^i(\boldsymbol{w}_{\ell'}, \boldsymbol{w}_{\ell'}, \boldsymbol{\varpi}_{\ell'}) = L^i(\boldsymbol{\varpi}_{\ell'}). \tag{4.21}$$

Using (4.16)–(4.20) we can pass to the limit in (4.21) and we obtain that

$$a^i(\boldsymbol{w}^i, \boldsymbol{\varpi}) + b^i(\boldsymbol{w}^i, \boldsymbol{w}^i, \boldsymbol{\varpi}) = L^i(\boldsymbol{\varpi}) \qquad \forall \boldsymbol{\varpi} \in V,$$

i.e. (4.2) holds, which completes the proof. ■

4.2 A priori estimates

We now ascertain a priori estimates for $\boldsymbol{u}^i$, q^i, u^i for $i = 1, 2, \ldots, n$. In the first step, we test (4.1) with $\boldsymbol{w}^i = (\boldsymbol{u}^i, q^i, Eu^i)$ and sum over $i = 1, 2, \ldots, r$ for some

$r \leq n$. We first focus on some terms which are used by the derivation of the a priori estimates.

$$2\sum_{i=1}^{r}\int_D \left(h^i \boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}\right)\cdot \boldsymbol{u}^i \, dy = \int_D h^r \left|\boldsymbol{u}^r\right|^2 dy$$
$$+\sum_{i=1}^{r}\int_D \left\{ \frac{1}{h^i}\left|h^i\boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}\right|^2 + \frac{h^{i-1}}{h^i}\left(h^i - h^{i-1}\right)\left|\boldsymbol{u}^{i-1}\right|^2 \right\} dy,$$

$$-2\int_D \left(\frac{h^i - h^{i-1}}{\Delta t}\right)\frac{\partial (y_2 \boldsymbol{u}^i)}{\partial y_2}\cdot \boldsymbol{u}^i \, dy$$
$$= \int_0^L \left(\frac{h^i - h^{i-1}}{\Delta t}\right)\left|u_2^i\right|^2 (y_1, 1)\, dy_1 - \int_D \left(\frac{h^i - h^{i-1}}{\Delta t}\right)\left|\boldsymbol{u}^i\right|^2 dy,$$

$$2\sum_{i=1}^{r}\int_0^L \left(u^i - u^{i-1}\right) u^i \, dy_1 = \int_0^L \left|u^r\right|^2 dy_1 + \sum_{i=1}^{r}\int_0^L \left|u^i - u^{i-1}\right|^2 dy_1,$$

$$U^0 \equiv 0, \qquad U^i \equiv \sum_{k=1}^{i} u^k \Delta t, \qquad \frac{U^i - U^{i-1}}{\Delta t} = u^i, \tag{4.22}$$

$$a\sum_{i=1}^{r}\int_0^L \frac{\partial U^i}{\partial y_1}\frac{\partial u^i}{\partial y_1}\, dy_1 \, \Delta t$$
$$= \frac{a}{2}\left\{ \int_0^L \left|\frac{\partial U^r}{\partial y_1}\right|^2 dy_1 + \sum_{i=1}^{r}\int_0^L \left|\frac{\partial (U^i - U^{i-1})}{\partial y_1}\right|^2 dy_1 \right\},$$

$$b\sum_{i=1}^{r}\int_0^L U^i u^i \, dy_1 \, \Delta t = \frac{b}{2}\left\{ \int_0^L \left|U^r\right|^2 dy_1 + \sum_{i=1}^{r}\int_0^L \left|U^i - U^{i-1}\right|^2 dy_1 \right\},$$

and finally, let us recall (4.11), i.e.

$$b^i(\boldsymbol{w}^i, \boldsymbol{w}^i, \boldsymbol{w}^i) = 0$$

for $\boldsymbol{w}^i = (\boldsymbol{u}^i, \cdot, \cdot)$. Now, (4.1) and (4.22) easily yield

$$
\begin{aligned}
& \int_D h^r \left(|\boldsymbol{u}^r|^2 + \varepsilon \, |q^r|^2 \right) \, dy + E \int_0^L |u^r|^2 \, dy_1 \\
& + \sum_{i=1}^r \int_D \frac{1}{h^i} \left(\left| h^i \boldsymbol{u}^i - h^{i-1} \boldsymbol{u}^{i-1} \right|^2 \; + \; \varepsilon \left| h^i q^i - h^{i-1} q^{i-1} \right|^2 \right) \, dy \\
& + \sum_{i=1}^r E \int_0^L \left| u^i - u^{i-1} \right|^2 dy_1 \\
& + 2 \sum_{i=1}^r \left(\sum_{m=1}^2 \nu a_1^i(u_m^i, u_m^i) + \varepsilon a_1^i(q^i, q^i) dy \right) \Delta t \\
& + 2 \sum_{i=1}^r \int_0^L \left\{ (1-\lambda)\kappa \left(u_2^i - \Upsilon h^i \right) \left(u_2^i - u^i \right) + \lambda \kappa \left| u_2^i - u^i \right|^2 \right\} \, dy_1 \, \Delta t \\
& + 2c \sum_{i=1}^r \int_0^L \left| \frac{\partial u^i}{\partial y_1} \right|^2 \, dy_1 \, \Delta t \\
& + a \int_0^L \left\{ \left| \frac{\partial U^r}{\partial y_1} \right|^2 dy_1 + \sum_{i=1}^r \left| \frac{\partial (U^i - U^{i-1})}{\partial y_1} \right|^2 \right\} dy_1 \\
& + b \int_0^L \left\{ |U^r|^2 \, dy_1 + \sum_{i=1}^r \int_0^L \left| U^i - U^{i-1} \right|^2 \right\} dy_1 \\
= \; & - \sum_{i=1}^r \int_D \frac{h^{i-1}}{h^i} \left(\Upsilon h^i \right) \left(\left| \boldsymbol{u}^{i-1} \right|^2 + \varepsilon \left| q^{i-1} \right|^2 \right) dy \, \Delta t \\
& + 2 \sum_{i=1}^r \int_0^1 \left(q_{in}^i(y_2) \, u_1^i(0, y_2) - q_{out}^i(y_2) \, u_1^i(L, y_2) \right) \, \ell \, dy_2 \, \Delta t \\
& + 2 \sum_{i=1}^r \int_0^L q_w^i(y_1) \, u_2^i(y_1, 1) \, dy_1 \, \Delta t,
\end{aligned}
$$

whereby Υh^i denotes the time difference

$$
\Upsilon h^i \equiv \frac{h^i - h^{i-1}}{\Delta t}.
$$

After neglecting positive terms and with the assistance of Lemma 4.2 from the last equality, we obtain

$$
\begin{aligned}
&\int_D h^r \left(|\boldsymbol{u}^r|^2 + \varepsilon\,|q^r|^2\right) dy + \int_0^L E\,|u^r|^2\,dy_1 \\
&+\frac{2\alpha}{2+K^2}\sum_{i=1}^r \int_D \left(\nu\left|\nabla \boldsymbol{u}^i\right|^2 + \varepsilon\left|\nabla q^i\right|^2\right) dy\Delta t \\
&+2c\sum_{i=1}^r \int_0^L \left|\frac{\partial u^i}{\partial y_1}\right|^2 dy_1\;\Delta t + a\int_0^L \left|\frac{\partial U^r}{\partial y_1}\right|^2 dy_1 + b\int_0^L |U^r|^2\,dy_1 \\
&\quad\le \sum_{i=1}^r \max_{0\le y_1\le L}\frac{h^{i-1}}{(h^i)^2}\left[-\left(\Upsilon h^i\right)\right]_+ \int_D h^i\left(|\boldsymbol{u}^i|^2+\varepsilon|q^i|^2\right) dy\;\Delta t \\
&\qquad + C_1\sum_{i=1}^r\left(\int_D \left|\nabla \boldsymbol{u}^i\right|^2 dy\right)^{1/2}\left[\left\|q^i_{in}\right\|_{L^2(0,1)} + \left\|q^i_{out}\right\|_{L^2(0,1)} + \left\|q^i_{ext}\right\|_{L^2(0,L)}\right]\Delta t \\
&\qquad + 2(1-\lambda)\kappa\sum_{i=1}^r\int_0^L \left|\left(u_2^i - \Upsilon h^i\right)\left(u_2^i - u^i\right)\right| dy_1\Delta t,
\end{aligned}
$$

where C_1 depends only on D. Furthermore, by using Hölder's inequality and Sobolev's embeddings (Propositions 3.1 and 3.2) on the right-hand side, we have

$$
\begin{aligned}
&\int_D h^r \left(|\boldsymbol{u}^r|^2 + \varepsilon\,|q^r|^2\right) dy + \int_0^L E\,|u^r|^2\,dy_1 \\
&+\frac{2\alpha\nu}{2+K^2}\sum_{i=1}^r \int_D \left(\left|\nabla \boldsymbol{u}^i\right|^2 + \frac{\varepsilon}{\nu}\left|\nabla q^i\right|^2\right) dy\Delta t + 2c\sum_{i=1}^r\int_0^L \left|\frac{\partial u^i}{\partial y_1}\right|^2 dy_1\;\Delta t \\
&\quad\le \sum_{i=1}^r\left(H_n^i\int_D h^i\left(|\boldsymbol{u}^i|^2+\varepsilon|q^i|^2\right) dy\; + \frac{(1-\lambda)\kappa}{2}\int_0^L |u^i|^2 dy_1\right)\Delta t \\
&\qquad + C_2\left(\sum_{i=1}^r\int_D\left|\nabla \boldsymbol{u}^i\right|^2 dy\;\Delta t\right)^{\frac12}\left[\left(\sum_{i=1}^r\left\|q^i_{\partial D}\right\|^2\Delta t\right)^{\frac12}\right. \\
&\qquad\qquad \left. + (1-\lambda)\kappa\left(\sum_{i=1}^r\|\Upsilon h^i\|^2\Delta t\right)^{\frac12} + (1-\lambda)\kappa\left(\sum_{i=1}^r\|\boldsymbol{u}^i\|^2_{L^2 D}\Delta t\right)^{\frac12}\right] \\
&\qquad + (1-\lambda)\kappa C_3\left(\sum_{i=1}^r\int_0^L\left|\frac{\partial u^i}{\partial y_1}\right|^2\;\Delta t\right)^{\frac12}\left(\sum_{i=1}^r\left\|\Upsilon h^i\right\|^2\Delta t\right)^{\frac12}
\end{aligned}
$$

where

$$
\left\|q^i_{\partial D}\right\|^2 \equiv \left\|q^i_{in}\right\|^2_{L^2(0,1)} + \left\|q^i_{out}\right\|^2_{L^2(0,1)} + \left\|q^i_{ext}\right\|^2_{L^2(0,L)},\quad \left\|\Upsilon h^i\right\| \equiv \left\|\Upsilon h^i\right\|_{L^2(0,L)}
$$

and

$$H_n^i \equiv \max_{0\le y_1 \le L}\left[-\frac{h^{i-1}}{(h^i)^2}\left(\Upsilon h^i\right)(y_1)\right]_+ .$$

Applying Young's inequality to the right-hand side one easily gets

$$\zeta_n(t) \le R_n \int_0^t \zeta_n(s)ds + M\int_0^t f_n(s)ds, \tag{4.23}$$

where

$$\zeta_n(t) = \int_D h^i\left(\left|\boldsymbol{u}^i\right|^2 + \varepsilon\left|q^i\right|^2\right)dy + \int_0^L E|u|^2 dy_1 \qquad \text{for} \quad t\in((i-1)\Delta t, i\Delta t]$$

$i = 0, 1, 2, \ldots, n,$

$$R_n \equiv H_n + \frac{(1-\lambda)\kappa}{2E} + (1-\lambda)\kappa^2\frac{(2+K^2)C_2^2}{2\nu},$$

$$M \equiv \frac{(2+K^2)C_2^2}{2\alpha\nu}\left(\kappa^2(1-\lambda)+1\right) + \frac{C_3^2\kappa^2(1-\lambda)}{4c},$$

$$H_n \equiv \max_i H_n^i$$

and

$$f_n(t) = \left\|q^i_{\partial D}\right\|^2 + (1-\lambda)\left\|\Upsilon h^i\right\|^2 \qquad \text{for} \quad t\in((i-1)\Delta t, i\Delta t],\ i=0,1,2,\ldots,n.$$

Next, by applying Gronwall's lemma in (4.23), see (4.80) and (4.81) below we obtain

$$\zeta_n(t) \le M\int_0^t f_n(s)ds\, e^{R_n t}$$

for almost all $t\in[0,T]$, and the first part of the following theorem follows easily.

Remark 4.1 (κ approximation). Note that if $\lambda = 1$, constants M, R_n do not depend on κ. Thus, the following a priori estimate (4.24) yields that if $\kappa \to \infty$, we formally obtain that $u_2 = u$, i.e. $v_2(x_1, h(x_1,t), t) = \frac{\partial \eta}{\partial t}(x_1, t)$ the domain deformation h is kept fixed). We do not prove, however, the convergence of $\boldsymbol{u}_\kappa \to \boldsymbol{u}$ in this dissertation. $\diamond$

Theorem 4.2 (A priori estimates)**.** *Under the assumptions (3.15), the following a priori estimates hold:*

$$
\begin{aligned}
&\max_{1\le r\le n}\left[\int_D h^r\left(|\boldsymbol{u}^r|^2+\varepsilon\,|q^r|^2\right)\,dy+\int_0^L E\,|u^r|^2\,dy_1\right]\\
&+\max_{1\le r\le n}\left[a\int_0^L\left|\frac{\partial U^r}{\partial y_1}\right|^2 dy_1+b\int_0^L |U^r|^2\,dy_1\right]\\
&+\sum_{i=1}^n\int_D\frac{1}{h^i}\left[\left|h^i\boldsymbol{u}^i-h^{i-1}\boldsymbol{u}^{i-1}\right|^2+\varepsilon\left|h^iq^i-h^{i-1}q^{i-1}\right|^2\right]\,dy\\
&+\sum_{i=1}^n\left(\int_0^L E\left|u^i-u^{i-1}\right|^2+2\lambda\kappa\left|u_2^i(y_1,1)-u^i(y_1)\right|^2\Delta t\right)\,dy_1 \qquad (4.24)\\
&+\frac{\alpha\nu}{2+K^2}\sum_{i=1}^n\int_D\left(\left|\nabla\boldsymbol{u}^i\right|^2+\frac{\varepsilon}{\nu}\left|\nabla q^i\right|^2\right)\,dy\,\Delta t\\
&+\sum_{i=1}^n\int_0^L c\left|\frac{\partial u^i}{\partial y_1}\right|^2(y_1)\,dy_1\,\Delta t\\
&\le P\sum_{i=1}^n\left(\left\|q^i_{\partial D}\right\|^2+(1-\lambda)\left\|\Upsilon h^i\right\|^2_{L^2(0,L)}\right)\Delta t,
\end{aligned}
$$

where $P=Me^{R_nT}$, $T\equiv n\Delta t$ and $M,\ R_n,\ H_n$ are given above.

Note that

$$
H_n\longrightarrow\max_{0\le y_1\le L,0\le t\le T}\left[-\frac{1}{h(y_1,t)}\frac{\partial h}{\partial t}(y_1,t)\right]_+\quad\textit{as } n\to 0. \qquad (4.25)
$$

Moreover,

$$
\begin{aligned}
&\frac{E}{2}\sum_{i=1}^n\int_0^L\left|\frac{u^i-u^{i-1}}{\Delta t}\right|^2 dy_1\Delta t\\
&+\frac{cE}{4}\int_0^L\left\{2\left|\frac{\partial u^r}{\partial y_1}\right|^2+\sum_{i=1}^n\left|\frac{\partial\left(u^i-u^{i-1}\right)}{\partial y_1}\right|^2\right\}dy_1 \qquad (4.26)\\
&\qquad\qquad\le C\sum_{i=1}^n\left(\left\|q^i_{\partial D}\right\|^2+(1-\lambda)\left\|\Upsilon h^i\right\|^2_{L^2(0,L)}\right)\Delta t,
\end{aligned}
$$

where C depends on $M,\ R_n,\ H_n,\ a,\ b,\ c,\ E,\ T$.

Proof of Theorem 4.2. To prove (4.26), we test (4.1) with $\left(\mathbf{0},0,E\Upsilon u^i\right)$, where $\Upsilon u^i\equiv$

$(u^i - u^{i-1})/\Delta t$. Then we sum over $i = 1, 2, \ldots, r$, multiply by Δt and obtain

$$\int_0^L \left\{ E \sum_{i=1}^r \left|\Upsilon u^i\right|^2 dy_1 \Delta t + \frac{cE}{2}\left|\frac{\partial u^r}{\partial y_1}\right|^2 + \frac{cE}{2}\sum_{i=1}^r \left|\frac{\partial u^i}{\partial y_1} - \frac{\partial u^{i-1}}{\partial y_1}\right|^2 \right\} dy_1$$
$$= \int_0^L \left\{ -bE\sum_{i=1}^r U^i(u^i - u^{i-1}) - aE\sum_{i=1}^r \frac{\partial U^i}{\partial y_1}\frac{\partial (u^i - u^{i-1})}{\partial y_1} \right. \tag{4.27}$$
$$\left. + \kappa \sum_{i=1}^r \left[\lambda\left(u_2^i(y_1,1) - u^i(y_1)\right) + (1-\lambda)\left(u_2^i(y_1,1) - \Upsilon h^i(y_1)\right)\right] \Upsilon u^i \, \Delta t \right\} dy_1,$$

where we have used the third relation of (4.22). Using the discrete per partes, i.e. $\sum_{i=1}^r U^i(u^i - u^{i-1}) = U^r u^r - \sum_{i=1}^{r-1} u^{i+1}u^i\Delta t$, the first two terms on the right-hand side of (4.27) are equal to

$$\int_0^L \left\{ -bEU^r u^r + bE\sum_{i=1}^{r-1} u^{i+1}u^i \Delta t - aE\frac{\partial U^r}{\partial y_1}\frac{\partial u^r}{\partial y_1} + aE\sum_{i=1}^{r-1}\frac{\partial u^{i+1}}{\partial y_1}\frac{\partial u^i}{\partial y_1}\Delta t \right\} dy_1$$

Finally, with use of (4.28) and Young's inequality we estimate the right-hand side of (4.27) as

$$\int_0^L \left(\frac{bE}{2}\left(|U^r|^2 + |u^r|^2\right) + \frac{a^2E}{4c}\left|\frac{\partial U^r}{\partial y_1}\right|^2 + \frac{cE}{4}\left|\frac{\partial u^r}{\partial y_1}\right|^2 + \sum_{i=1}^r \left\{ \left[bE\left|u^i\right|^2 + aE\left|\frac{\partial u^i}{\partial y_1}\right|^2 \right.\right.\right.$$
$$\left.\left.\left. + \frac{\kappa^2\lambda^2}{2E}\left|u_2^i(y_1,1) - u^i(y_1)\right|^2 + \frac{\kappa^2(1-\lambda)^2}{E}\left(\left|\boldsymbol{u}^i\right|^2 + \left|\Upsilon h^i\right|^2\right) + \frac{E}{2}\left|\Upsilon u^i\right|^2\right] \Delta t \right\}\right) dy_1$$

Thus the assertion (4.26) of the theorem follows easily from the estimate (4.24) above. ■

In the sequel, the following estimate will be essential to get a priori estimate in the time variable t, and will result in equicontinuity in time variable of piecewise constant function, see the definition of $\boldsymbol{U}_n^s(t)$ (4.36) below.

Theorem 4.3 (Equicontinuity in time variable)**.** *A non-negative constant C exists such that*

$$\sum_{i=1}^{n-k} \int_D \left(\left|h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i\right|^2 + \varepsilon\left|h^{i+k}q^{i+k} - h^i q^i\right|^2 \right) dy \, \Delta t \, \le \, Ck\Delta t \tag{4.28}$$

for any $1 \le k < n$. The constant C does not depend on k, n.

Proof of Theorem 4.3.
1. Recalling the definition of the weak solution, let us fix $i \in \{1, 2, \ldots, n-k\}$ in (4.1), multiply by Δt and add (4.1) through $j = i+1, \ldots, i+k$ for fix test functions

$$\boldsymbol{\omega} = h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i, \quad v = h^{i+k}q^{i+k} - h^i q^i, \quad \vartheta = 0.$$

Then we sum the equality over $i = 1, 2, \ldots, n-k$, multiply by Δt and arrive at

$$\begin{aligned}
&\sum_{i=1}^{n-k}\int_D \left(\left|h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i\right|^2 + \varepsilon\left|h^{i+k}q^{i+k} - h^i q^i\right|^2\right)\, dy\,\Delta t \\
= \quad & (\Delta t)^2 \sum_{i=1}^{n-k}\sum_{j=i+1}^{i+k}\Bigg\{\int_D \Upsilon h^j \frac{\partial}{\partial y_2}\Big(y_2\boldsymbol{u}^j\Big)\cdot(h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i)\, dy \\
& -B^j(\boldsymbol{u}^j, \boldsymbol{u}^j, h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i) \\
& -\int_D \frac{h^j}{2}\boldsymbol{u}^j \mathrm{div}_j(\boldsymbol{u}^j)\cdot(h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i)\, dy \\
& -\nu\sum_{m=1}^{2} a_1(u^j_m, h^{i+k}u^{i+k}_m - h^i u^i_m) \qquad\qquad (4.29)\\
& +\int_D h^j g^j \mathrm{div}_h(h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i)\, dy - \varepsilon a_1(q^j, h^{i+k}q^{i+k} - h^i q^i) \\
& +\varepsilon\int_D \Upsilon h^j \frac{\partial}{\partial y_2}\Big(y_2 q^j\Big)(h^{i+k}q^{i+k} - h^i q^i)\, dy \\
& +\int_D h^j \mathrm{div}_j(\boldsymbol{u}^j)(h^{i+k}q^{i+k} - h^i q^i)\, dy \\
& -\int_0^1 \ell(q^j_{out} - \frac{1}{2}|u^j_1|^2)(h^{i+k}u^{i+k}_1 - h^i u^i_1)\,(L, y_2)\, dy_2 \\
& +\int_0^1 \ell(q^j_{in} - \frac{1}{2}|u^j_1|^2)(h^{i+k}u^{i+k}_1 - h^i u^i_1)\,(0, y_2)\, dy_2 \\
& +\int_0^L \Big(q^j_w - \frac{1}{2}u^j_2(u^j_2 - \Upsilon h^j)\Big)(h^{i+k}u^{i+k}_2 - h^i u^i_2)\,(y_1, 1)\, dy_1 \\
& +\int_0^L \Big(\lambda\kappa(u^j_2 - u^j) + (1-\lambda)\kappa(u^j_2 - \Upsilon h^j)\Big)(h^{i+k}u^{i+k}_2 - h^i u^i_2)\,(y_1, 1)\, dy_1 \\
& -\frac{\varepsilon}{2}\int_0^L \Upsilon h^j q^j(h^{i+k}q^{i+k} - h^i q^i)\,(y_1, 1)\, dy_1 \Bigg\}.
\end{aligned}$$

We will choose two most difficult terms on the right-hand side of the above equality and we will show that they can be estimated by $Ck/\Delta t$, where C does not depend on i, j, k, n. The other terms can be then estimated in a similar fashion.

2. Let us first recall (4.12), i.e.

$$B^j(\boldsymbol{u}^j,\boldsymbol{u}^j,h^{i+k}\boldsymbol{u}^{i+k}-h^i\boldsymbol{u}^i)$$
$$\equiv \int_D \left(h^j u_1^j \left(\frac{\partial \boldsymbol{u}^j}{\partial y_1} - \frac{y_2}{h^j}\frac{\partial h^j}{\partial y_1}\frac{\partial \boldsymbol{u}^j}{\partial y_2}\right) + u_2^j \frac{\partial \boldsymbol{u}^j}{\partial y_2}\right)\cdot \left(h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i\right)\, dy$$

and focus on the nonlinear term

$$\int_D u_1^j y_2 \frac{\partial h^j}{\partial y_1}\frac{\partial \boldsymbol{u}^j}{\partial y_2}\cdot\left(h^{i+k}\boldsymbol{u}^{i+k}\right)\, dy \tag{4.30}$$

that we shall treat in all details. Going back to the item 1., the corresponding sum can be estimated by

$$\begin{aligned}
&\sum_{i=1}^{n-k}\left(\int_D \left|\sum_{j=i+1}^{i+k}|u_1^j|\left|\frac{\partial \boldsymbol{u}^j}{\partial y_2}\right|\right|^{6/5} dy\right)^{5/6}\left(\int_D \left|\boldsymbol{u}^{i+k}\right|^6 dy\right)^{1/6}\frac{K}{\alpha}(\Delta t)^2\\
&\le \left[\sum_{i=1}^{n-k}\left(\int_D \left|\sum_{j=i+1}^{i+k}|u_1^j|\left|\frac{\partial \boldsymbol{u}^j}{\partial y_2}\right|\right|^{6/5} dy\right)^{5/4}\Delta t\right]^{2/3} \\
&\qquad \times\left[\sum_{i=1}^{n}\left(\int_D \left|\boldsymbol{u}^{i+k}\right|^6 dy\right)^{1/2}\Delta t\right]^{1/3}\frac{K}{\alpha}\Delta t.
\end{aligned} \tag{4.31}$$

We can further estimate the first term on the right-hand side of (4.31) using Minkowski's inequality in $L^{6/5}(D)$, discrete Hölder's inequality for $p=3/2,\ q=3/1$, then by applying Hölder's inequality for $p=5/3,\ q=5/2$ and then again by discrete Hölder's inequality for $p=4/3,\ q=4/1$. We obtain

$$\begin{aligned}
&\left[\sum_{i=1}^{n-k}\left(\sum_{j=i+1}^{i+k}\left(\int_D |u_1^j|^{6/5}\left|\frac{\partial \boldsymbol{u}^j}{\partial y_2}\right|^{6/5} dy\right)^{5/6}\right)^{3/2}\Delta t\right]^{2/3}\\
&\le \left[\sum_{i=1}^{n-k}\sum_{j=i+1}^{i+k}\left(\int_D |u_1^j|^{6/5}\left|\frac{\partial \boldsymbol{u}^j}{\partial y_2}\right|^{6/5} dy\right)^{5/4} k^{1/2}\Delta t\right]^{2/3}\\
&\le \left[\sum_{i=1}^{n}\left(\int_D \left|\frac{\partial \boldsymbol{u}^i}{\partial y_2}\right|^{2} dy\right)^{3/4}\left(\int_D |u_1^i|^3 dy\right)^{1/2}\Delta t\right]^{2/3} k\\
&\le \left[\sum_{i=1}^{n}\left(\int_D \left|\nabla \boldsymbol{u}^i\right|^2 dy\right)\Delta t\right]^{1/2}\left[\sum_{i=1}^{n}\left(\int_D |u_1^i|^3 dy\right)^2\Delta t\right]^{1/6} k
\end{aligned} \tag{4.32}$$

Thus, (4.31) can be estimated with the assistance of (4.32) by

$$\|\boldsymbol{u}_n\|_{L^2(0,T;(H^1(D))^2)} \|\boldsymbol{u}_n\|_{L^6(0,T;(L^3(D))^2)} \|\boldsymbol{u}_n\|_{L^3(0,T;(L^6(D))^2)} \frac{K}{\alpha} k\Delta t, \tag{4.33}$$

where we have defined the step function $\boldsymbol{u}_n : \ [0,T] \longrightarrow \boldsymbol{V}$ for $n \in N$ such that $\boldsymbol{u}_n(t) = \boldsymbol{u}^i \quad$ for $\ t \in ((i-1)\Delta t, i\Delta t]\ ,\ i = 0,1,\dots,n,\ \boldsymbol{u}^0 = \boldsymbol{0}$, see Section 6 below. The a priori estimates from Theorem 4.2 yield the boundedness of $\{\boldsymbol{u}_n\}_{n=1}^{\infty}$ with respect to n in the space $L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;(L^2(D))^2)$ and Proposition 3.1 gives its boundedness in the space

$$L^{2p/(p-2)}(0,T;(L^p(D))^2) \quad \text{for any} \quad p \geq 2,$$

i.e. also for $p = 3, 6$.

3. As for boundary terms, let us estimate the nonlinear term by applying Hölder's inequality ($p = q = 2$) and then using two times discrete Hölder's inequality for $p = 4,\ q = 4/3$

$$\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{n-k}\sum_{j=i+1}^{i+k}\int_0^L \left|u_2^j\right|^2 h^{i+k} u_2^{i+k}(y_1,1)\, dy_1 (\Delta t)^2 \\
&\leq \alpha^{-1}\sum_{i=1}^{n-k}\left(\int_0^L \left|u_2^{i+k}\right|^2 (y_1,1)dy_1\right)^{1/2}\left[\sum_{j=i+1}^{i+k}\left(\int_0^L \left|u_2^j\right|^4 (y_1,1)dy_1\right)^{1/2}\right](\Delta t)^2 \\
&\leq \alpha^{-1}\left(\sum_{i=1}^{n}\left(\int_0^L \left|u_2^i\right|^2 (y_1,1)dy_1\right)^2 \Delta t\right)^{1/4} \qquad\qquad (4.34)\\
&\qquad \times\left(\sum_{i=1}^{n-k}\left[\sum_{j=i+1}^{i+k}\left(\int_0^L \left|u_2^j\right|^4 (y_1,1)dy_1\right)^{1/2}\right]^{4/3}\Delta t\right)^{3/4}\Delta t \\
&\leq \alpha^{-1}\|\boldsymbol{u}_n\|_{L^4(0,T;(L^2(S))^2)}\left(\sum_{i=1}^{n-k}\sum_{j=i+1}^{i+k}\left(\int_0^L \left|u_2^j\right|^4 (y_1,1)dy_1\right)^{2/3} k^{1/3}\Delta t\right)^{3/4}\Delta t \\
&\leq \alpha^{-1}\|\boldsymbol{u}_n\|_{L^4(0,T;(L^2(S))^2)}\|\boldsymbol{u}_n\|^2_{L^{8/3}(0,T;(L^4(S))^2)} k\Delta t.
\end{aligned}$$

In this case, the boundedness of $\{\boldsymbol{u}_n\}_{n=1}^{\infty}$ with respect to n in the space $L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;L^2(D)^2)$ and Proposition 3.2 yield its boundedness in the spaces

$$L^{2p/(p-1)}(0,T;(L^p(S))^2) \quad \text{for any} \quad p \geq 2,$$

i.e. also for $p = 2, 4$. Direct calculations with all other terms, the details of which we omit, lead to (4.28). ■

Remark 4.2. Note that the test function used in previous proof, $\boldsymbol{\omega} = h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i$, does not fulfill the discrete divergence-free condition $\mathrm{div}_i\ \boldsymbol{\omega} = 0$. It means that $\mathrm{div}_i\ \boldsymbol{u}^i = 0$ and $\mathrm{div}_{i+k}\ \boldsymbol{u}^{i+k} = 0$ does not imply

$$\mathrm{div}_i\ \boldsymbol{\omega} = \frac{\partial\omega_1}{\partial y_1} - \frac{y_2}{h^i}\frac{\partial h^i}{\partial y_1}\frac{\partial\omega_1}{\partial y_2} + \frac{1}{h^i}\frac{\partial\omega_2}{\partial y_2} = 0,$$

since $h^i \equiv h(y_1, i\Delta t) \neq h^{i+k} \equiv h(y_1,(i+k)\Delta t)$ in general. This is the reason for the regularisation of the incompressibility condition (1.1) of the original mathematical model, see (2.26). ◇

4.3 Existence for the unsteady problem

Let us first construct sequences of approximate piecewise constant and piecewise linear functions

$$\boldsymbol{u}_n^s,\ \boldsymbol{U}_n^s,\ \boldsymbol{U}_n :\ [0,T] \longrightarrow \boldsymbol{V}$$

for $n \in N$ such that

$$\boldsymbol{u}_n^s(t) = \boldsymbol{u}^i \qquad \text{for} \quad t \in ((i-1)\Delta t, i\Delta t], \tag{4.35}$$

$i = 0,1,\dots,n,\ \boldsymbol{u}^0 = \boldsymbol{0}$,

$$\boldsymbol{U}_n^s(t) = \boldsymbol{u}^i h^i \qquad \text{for} \quad t \in ((i-1)\Delta t, i\Delta t], \tag{4.36}$$

and

$$\boldsymbol{U}_n(t) = \boldsymbol{u}^{i-1}h^{i-1} + \frac{t-(i-1)\Delta t}{\Delta t}\left(\boldsymbol{u}^i h^i - \boldsymbol{u}^{i-1}h^{i-1}\right) \tag{4.37}$$

for $t \in [(i-1)\Delta t, i\Delta t]$, $i = 1,\dots,n$. We analogically construct approximate functions for q, u and h, respectively,

$$q_n^s,\ Q_n^s,\ Q_n :\ [0,T] \longrightarrow H^1(D),$$

where $n \in N$ such that

$$q_n^s(t) = q^i \qquad \text{for} \quad t \in ((i-1)\Delta t, i\Delta t], \tag{4.38}$$

$i = 0,1,\dots,n,\ q(0) = 0$,

$$Q_n^s(t) = q^i h^i \qquad \text{for} \quad t \in ((i-1)\Delta t, i\Delta t], \tag{4.39}$$

$$Q_n(t) = q^{i-1}h^{i-1} + \frac{t-(i-1)\Delta t}{\Delta t}\left(q^i h^i - q^{i-1}h^{i-1}\right) \tag{4.40}$$

for $t \in [(i-1)\Delta t, i\Delta t]$, $i = 1,\dots,n$,

$$u_n^s,\ u_n\ :\ [0,T] \longrightarrow H_0^1(0,L),$$

$n \in N$ such that

$$u_n^s(t) = u^i \qquad \text{for} \quad t \in ((i-1)\Delta t, i\Delta t], \tag{4.41}$$

$i = 0, 1, \ldots, n,$

$$u_n(t) = u^{i-1} + \frac{t-(i-1)\Delta t}{\Delta t}\left(u^i - u^{i-1}\right) \quad \text{for} \quad t \in [(i-1)\Delta t, i\Delta t]$$

and finally,

$$h_n^s,\ h_n:\ [0,T] \longrightarrow H_0^1(0,L),$$

$n \in N$ such that

$$h_n^s(t) = h^i \qquad \text{for} \quad t \in ((i-1)\Delta t, i\Delta t], \tag{4.42}$$

$i = 0, 1, \ldots, n,$

$$h_n(t) = h^{i-1} + \frac{t-(i-1)\Delta t}{\Delta t}\left(h^i - h^{i-1}\right) \quad \text{for} \quad t \in [(i-1)\Delta t, i\Delta t].$$

Our plan is to pass $n \longrightarrow \infty$. Therefore, we need estimates which are uniform in n. According to the apriori estimates (4.24) and (4.26), we observe that the sequences

$$\begin{aligned} &\{\boldsymbol{u}_n^s\}_{n=1}^\infty,\ \{\boldsymbol{U}_n^s\}_{n=1}^\infty,\ \{\boldsymbol{U}_n\}_{n=1}^\infty \\ &\text{are bounded in} \\ &L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;L^2(D)^2). \end{aligned} \tag{4.43}$$

Since $|\boldsymbol{U}_n^s(x,t) - \boldsymbol{U}_n(x,t)| = \left|\boldsymbol{u}^i h^i - \boldsymbol{u}^{i-1}h^{i-1}\right| \left|\frac{t-i\Delta t}{\Delta t}\right| \le \left|\boldsymbol{u}^i h^i - \boldsymbol{u}^{i-1}h^{i-1}\right|$, for $t \in ((i-1)\Delta t,\ i\Delta t)$ and

$$\left(\int_0^T \int_D |\boldsymbol{U}_n^s - \boldsymbol{U}_n|^2 \ dy\ dt\right)^{\frac{1}{2}} = \sqrt{\Delta t}\left(\sum_{i=1}^n \int_D \left|\boldsymbol{u}^i h^i - \boldsymbol{u}^{i-1}h^{i-1}\right|^2 \ dy\right)^{\frac{1}{2}},$$

with the assistance of (4.24) we obtain

$$\|\boldsymbol{U}_n^s - \boldsymbol{U}_n\|_{L^2(0,T;(L^2(D))^2)} \le C\sqrt{\Delta t}. \tag{4.44}$$

Testing (4.1) with $\boldsymbol{\varpi} = (\boldsymbol{\omega}, v, \vartheta)$ such that

$$\boldsymbol{\omega} \in L^2(0,T;\boldsymbol{V}) \cap L^4(0,T;L^4(D)^2), \quad v = 0, \quad \vartheta = 0,$$

we find that

$$\left\{\frac{\partial \boldsymbol{U}_n}{\partial t}\right\}_{n=1}^\infty \quad \text{is bounded in} \quad L^2(0,T;\boldsymbol{V}^*) + L^{4/3}(0,T;L^{4/3}(D)^2). \tag{4.45}$$

From the third a priori estimate (4.28) we obtain

$$\int_0^{T-k\Delta t} \int_D |\boldsymbol{U}_n^s(t+k\Delta t) - \boldsymbol{U}_n^s(t)|^2 \, dy\ dt \le C\, k\Delta t. \tag{4.46}$$

Using similar arguments, we obtain the following assertions

$$\{\sqrt{\varepsilon}\, q_n^s\}_{n=1}^{\infty},\ \{\sqrt{\varepsilon}\, Q_n^s\}_{n=1}^{\infty},\ \{\sqrt{\varepsilon}\, Q_n\}_{n=1}^{\infty} \text{ are bounded in } L^2(0,T;H^1(D)) \cap L^{\infty}(0,T;L^2(D)), \tag{4.47}$$

$$\left\| \sqrt{\varepsilon}\ (Q_n^s - Q_n) \right\|_{L^2(Q)} \le C\sqrt{\Delta t}, \tag{4.48}$$

$$\left\{ \sqrt{\varepsilon} \frac{\partial Q_n}{\partial t} \right\}_{n=1}^{\infty} \text{ is bounded in } L^2(0,T;H^{-1}(D)), \tag{4.49}$$

$$\sqrt{\varepsilon} \int_0^{T-k\Delta t} \int_D |Q_n^s(t+k\Delta t) - Q_n^s(t)|^2\, dy\, dt \le C\, k\Delta t \tag{4.50}$$

and finally,

$$\{u_n^s\}_{n=1}^{\infty}, \{u_n\}_{n=1}^{\infty} \text{ are bounded in } L^{\infty}(0,T;H_0^1(0,L)), \tag{4.51}$$

$$\|u_n^s - u_n\|_{L^2(0,T;L^2(0,L))} \le C\sqrt{\Delta t}. \tag{4.52}$$

From (4.26) follows that

$$\left\{ \frac{\partial u_n}{\partial t} \right\}_{n=1}^{\infty} \text{ is bounded in } L^2(0,T;L^2(0,L)). \tag{4.53}$$

Previous observations help us to prove the following lemma.

Lemma 4.3. *There exist a subsequence of $\{n\}_{n=1}^{\infty}$ and functions $(\boldsymbol{u}, q, u) \in L^2(0,T;V) \cap L^{\infty}(0,T;L^2(D)^2 \times L^2(D) \times L^2(0,L))$ (we denote the subsequence for simplicity again $\{n\}_{n=1}^{\infty}$), such that*

$$h_n \longrightarrow h \qquad \text{in } W^{1,\infty}(0,T;C([0,L])), \tag{4.54}$$

$$h_n^s \longrightarrow h \qquad \text{in } L^{\infty}(0,T;C^1([0,L])), \tag{4.55}$$

$$\begin{aligned} \boldsymbol{U}_n^s \longrightarrow h\boldsymbol{u}\ ,\quad \boldsymbol{u}_n^s \longrightarrow \boldsymbol{u} \qquad & \text{weakly in } L^2(0,T;\boldsymbol{V}), \\ & \text{strongly in } L^p(D\times(0,T))^2 \text{ for } 1 \le p < 4, \\ & \text{strongly in } L^2(0,T;L^p(S)^2)\ \forall p > 1, \end{aligned} \tag{4.56}$$

$$\begin{aligned} q_n^s \longrightarrow q \qquad & \text{weakly in } L^2(0,T;H^1(D)), \\ & \text{strongly in } L^2(D\times(0,T)), \end{aligned} \tag{4.57}$$

$$Q_n \longrightarrow hq \qquad \text{weakly in } H^1(0,T;H^{-1}(D)), \tag{4.58}$$

$$u_n \longrightarrow u \qquad \text{weakly in } H^1((0,L)\times(0,T)) \tag{4.59}$$

as $n \longrightarrow \infty$.

Proof of Lemma 4.3.
1. Since $h \in C^1([0,T] \times [0,L])$, (4.54) and (4.55) follow easily. Indeed,

$$\|h_n - h\|_{W^{1,\infty}(0,T;C[0,L])} =$$
$$\max_{1\le i\le n}\left[\max_{(i-1)\Delta t\le t\le i\Delta t}\left\{\max_{1<x_1<L}\left|\frac{1}{\Delta t}\int_{(i-1)\Delta t}^{i\Delta t}\left(\frac{\partial h}{\partial t}(x,s)-\frac{\partial h}{\partial t}(x,t)\right)ds\right|\right.\right.$$
$$+\max_{1<x_1<L}\left|\frac{t-(i-1)\Delta t}{\Delta t}\Big(h(x,i\Delta t)-h(x,t)\Big)\right.$$
$$\left.\left.\left.+\frac{i\Delta t-t}{\Delta t}\Big(h(x,(i-1)\Delta t)-h(x,t)\Big)\right|\right\}\right]$$
$$\longrightarrow 0 \ \text{ if } \ \Delta t = \frac{T}{n} \to 0.$$

$$\|h_n - h\|_{L^\infty(0,T;C^1[0,L])} =$$
$$\max_{1\le i\le n}\left[\max_{(i-1)\Delta t\le t\le i\Delta t}\left\{\max_{1<x_1<L}\left|\left(\frac{\partial h}{\partial y}(x,(i-1)\Delta t)-\frac{\partial h}{\partial y}(x,t)\right)ds\right|\right.\right.$$
$$\left.\left.+\max_{1<x_1<L}|h(x,(i-1)\Delta t)-h(x,t)|\right\}\right]$$
$$\longrightarrow 0 \ \text{ if } \ \Delta t = \frac{T}{n} \to 0.$$

2. The weak convergence (4.56) is the consequence of (4.43) and (4.54) as $\boldsymbol{U}_n^s = h_n^s \boldsymbol{u}_n^s$, see (4.35), (4.36) above.

To prove the strong convergence of $\boldsymbol{U}_n^s$ in L^p, choose k such that $k\Delta t < T$, $D^\delta = (\delta_1, L-\delta_1) \times (\delta_2, 1-\delta_2)$ for small $|\delta|$, for $\ell < \infty$ and

$$(\boldsymbol{U}_n^s)^\ell = \min\left\{1, \frac{\ell}{|\boldsymbol{U}_n^s|}\right\}\boldsymbol{U}_n^s.$$

The estimate (4.46) then yields

$$\int_0^{T-k\Delta t}\int_D \left|(\boldsymbol{U}_n^s)^\ell(y,t+k\Delta t)-(\boldsymbol{U}_n^s)^\ell(y,t)\right|\,dy\,dt \le C\sqrt{k\Delta t}. \tag{4.60}$$

We further claim

$$\int_0^T\int_D \left|(\boldsymbol{U}_n^s)^\ell(y+\delta,t)-(\boldsymbol{U}_n^s)^\ell(y,t)\right|\,dy\,dt \le C\ell|\delta|, \tag{4.61}$$

where, if necessary, we extend $\boldsymbol{U}_n^s$ outside of D by zero. Indeed, the estimate (4.43) implies

$$\begin{aligned}
&\int_0^T \int_{D^\delta} \left|\left(\boldsymbol{U}_n^s\right)^\ell (y+\delta,t) - \left(\boldsymbol{U}_n^s\right)^\ell (y,t)\right| \, dy \, dt \\
&= \int_0^T \int_{D^\delta} \left| \int_0^1 \frac{d\left(\boldsymbol{U}_n^s\right)^\ell}{ds}(s(y+\delta)+(1-s)y,t)\, ds\right| \, dy\, dt \\
&\le \int_0^T \int_{D^\delta} \int_0^1 \left| \sum_{i=1}^2 \frac{\partial \left(\boldsymbol{U}_n^s\right)^\ell}{\partial y_i}(y+s\delta,t)\delta_i \right| ds\, dy\, dt \\
&\le C \|\boldsymbol{U}_n^s\|_{L^2(0,T;(H^1D)^2)} |\delta|
\end{aligned}$$

and

$$\int_0^T \int_{D\setminus D^\delta} \left|\left(\boldsymbol{U}_n^s\right)^\ell (y+\delta,t) - \left(\boldsymbol{U}_n^s\right)^\ell (y,t)\right| \, dy \, dt \le C\ell|\delta|.$$

Thus for fixed ℓ we can conclude that the set $\{(\boldsymbol{U}_n^s)^\ell\}_{n=1}^\infty$ is precompact in $L^1(D \times (0,T))^2$, see Riesz's (Kolmogorov's) compactness criteria [KJF77, Theorem 2.13.1, p.88] or [AL83, Lemma 1.9]. Then using the inequality which we borrow from [AL83]

$$|\left(\boldsymbol{U}_n^s\right)^\ell - \boldsymbol{U}_n^s| \le \frac{1}{\ell} |\left(\boldsymbol{U}_n^s\right)|^2, \tag{4.62}$$

we obtain that also $\{\boldsymbol{U}_n^s\}_{n=1}^\infty$ is precompact in $L^1(D \times (0,T))^2$ *, therefore a subsequence exists which converges strongly in this space. Hence, we obtain strong convergence

$$\boldsymbol{U}_n^s \to h\boldsymbol{u} \text{ in } L^1(D \times (0,T))^2$$

as $n \to \infty$. According to Proposition 3.1 and the estimate (4.43) we observe that $\boldsymbol{U}_n^s$, $\boldsymbol{u}_n^s$ are bounded in $L^4(D \times (0,T))^2$. Due to the interpolation argument for p such that $1 \le p < 4$: $\|\boldsymbol{U}_n^s - h\boldsymbol{u}\|_{L^p}^p \le \|\boldsymbol{U}_n^s - h\boldsymbol{u}\|_{L^1}^{\frac{1}{3}(4-p)} \|\boldsymbol{U}_n^s - h\boldsymbol{u}\|_{L^4}^{\frac{4}{3}(p-1)}$, we obtain the strong convergence of $\boldsymbol{U}_n^s$ in the space stated in second assertion of (4.56).

*We prove the precompactness of $\{\boldsymbol{U}_n^s\}_{n=1}^\infty$ from (4.60) and the inequality (4.62). Hence

$$\begin{aligned}
\int_0^{T-k\Delta t} \int_D |\boldsymbol{U}_n^s(y,t+k\Delta t) - \boldsymbol{U}_n^s(y,t)| \;&\le\; \int_0^{T-k\Delta t} \int_D \left|\boldsymbol{U}_n^s(y,t+k\Delta t) - \left(\boldsymbol{U}_n^s\right)^\ell (y,t+k\Delta t)\right| \\
&+\; \int_0^{T-k\Delta t} \int_D \left|\left(\boldsymbol{U}_n^s\right)^\ell (y,t+k\Delta t) - \left(\boldsymbol{U}_n^s\right)^\ell (y,t)\right| \\
&+\; \int_0^{T-k\Delta t} \int_D \left|\left(\boldsymbol{U}_n^s\right)^\ell (y,t) - \boldsymbol{U}_n^s(y,t)\right| \\
&\le\; 2C\sqrt{k\Delta t} \quad \text{if } \ell \ge \frac{2}{C\sqrt{k\Delta t}} \|\boldsymbol{U}_n^s\|^2_{L^2(0,T;L^2(D)^2)}.
\end{aligned}$$

Analogically, we obtain the equicontinuity of $\boldsymbol{U}_n^s(y,t)$ in space variable from (4.61) and (4.62) if $\ell = \frac{2}{C\sqrt{|\delta|}}$.

Without a loss of generality, we consider $p \geq 2$ and on the base of Proposition 3.2 (3.21), we obtain

$$\begin{aligned}\|\boldsymbol{U}_n^s - h\boldsymbol{u}\|^2_{L^2(0,T;L^p(S)^2)} &= \int_0^T \|\boldsymbol{U}_n^s - h\boldsymbol{u}\|^2_{L^p(S)^2} \\ &\leq c\int_0^T \|\nabla(\boldsymbol{U}_n^s - h\boldsymbol{u})\|^{\frac{2(p-1)}{p}}_{L^2(D)^2} \|\boldsymbol{U}_n^s - h\boldsymbol{u}\|^{\frac{2}{p}}_{L^2(D)^2} \\ &\leq c\|\boldsymbol{U}_n^s - h\boldsymbol{u}\|^{\frac{2}{p}}_{L^2(D\times(0,T))^2} \|\boldsymbol{U}_n^s - h\boldsymbol{u}\|^{\frac{2(p-1)}{p}}_{L^2(0,T;H^1(D)^2)},\end{aligned}$$

hence we obtain the third assertion of (4.56).

3. To prove (4.57), use the estimates (4.47)–(4.48) for fixed $\varepsilon > 0$ and apply the above reasoning. We prove (4.58) for a fixed $\varepsilon > 0$ as follows. From (4.47), (4.48), (4.49) and the previous assertions (4.57) and (4.54), we obtain weak convergences

$$\begin{aligned} Q_n &\rightharpoonup hq \qquad \text{in } L^2(0,T;H^1(D)), \\ Q_n &\rightharpoonup \chi \qquad \text{in } L^2(0,T;H^{-1}(D)). \end{aligned}$$

Note that $\int_0^T \left\langle \frac{\partial Q_n}{\partial t}, \phi \right\rangle_{H^1(D)} dt$ in (4.64) is equal to $\int_0^T \int_D \frac{\partial Q_n}{\partial t}\phi$. Thus by passing $n \to \infty$ in (4.64) below, we obtain

$$-\int_0^T \langle \chi, \phi \rangle_{H^1(D)}\, dt = \int_0^T \int_D hq\frac{\partial \phi}{\partial t}\, dt$$

for $\phi \in L^2(0,T;H^1(D)) \cap H^{1,1}(0,T;L^2(D))$. Again, due to the pairing between $H^1(D)$ and $H^{-1}(D)$, the right-hand side of this equality can be written as

$$\int_0^T \left\langle hq, \frac{\partial \phi}{\partial t} \right\rangle_{H^1(D)} dt.$$

After putting $\phi(x,t) = w(x)\xi(t),\ w \in H^1(D),\ \xi \in C_0^\infty(0,T)$, we obtain from the identity (4.3) that

$$-\int_0^T \langle \chi, \phi \rangle_{H^1(D)}\, \xi(t) dt = \int_0^T \langle hq, w \rangle_{H^1(D)}\, \xi'(t) dt$$

for $w \in H^1(D),\ \xi \in C_0^\infty(0,T)$, and thus χ is the distributive derivate

$$\chi = \frac{\partial(hq)}{\partial t} \in L^2(0,T;H^{-1}(D)).$$

4. Finally, in order to prove (4.59) we use (4.51), which implies $u_n^s \rightharpoonup u^1,\ u_n \rightharpoonup u^2$ in $L^2(0,T;H_0^1(0,L))$. As a consequence of (4.52), we obtain that $u^1 = u^2$. Hence we obtain from (4.53) that $\frac{\partial u_n}{\partial t} \to \zeta$ weakly in $L^2(0,T;L^2(0,L))$. Since $\int_0^T \int_0^L \frac{\partial u_n}{\partial t}\xi = -\int_0^T \int_0^L u_n \frac{\partial \xi}{\partial t}$ by passing to the limit for $n \to \infty$, we obtain that $\zeta = \frac{\partial u}{\partial t}$. ■

Now, let us simply assume that the test functions $(\boldsymbol{\psi}, \phi, \xi)$ in (3.17) are more regular than required, i.e.

$$(\boldsymbol{\psi}, \phi, \xi) \in C([0, T]; V)\ , \qquad \boldsymbol{\psi} \in C^1(\overline{D} \times [0, T])^2. \tag{4.63}$$

We denote

$$\boldsymbol{\psi}^i(y) = \boldsymbol{\psi}(y, i\Delta t), \qquad \phi^i(y) = \phi(y, i\Delta t), \qquad \xi^i(y_1) = \xi(y_1, i\Delta t)$$

and construct sequences of functions

$$\boldsymbol{\psi}_n, \quad \boldsymbol{\psi}_n^s, \quad \phi_n^s, \quad \xi_n^s$$

in the same fashion as above. It is straightforward to verify that as $n \longrightarrow \infty$, then

$$\begin{aligned} \boldsymbol{\psi}_n \to \boldsymbol{\psi} \ \text{ in } H^1(D \times (0,T))^2, \ \ \boldsymbol{\psi}_n^s \to \boldsymbol{\psi} \ \text{ in } L^\infty(0, T; C^1(\overline{D})^2), \\ \phi_n^s \to \phi \ \text{ in } L^2(0, T; H^1(D)) \ \text{ and } \ \xi_n^s \to \xi \ \ L^\infty(0, T; H_0^1(0, L)). \end{aligned}$$

After these preparations it is now easy to prove the existence of a solution to our problem. Putting $(\boldsymbol{\psi}_n^s(i\Delta t), \phi_n^s(i\Delta t), \xi_n^s(i\Delta t))$ into (4.1), multiplying it by Δt and adding up through $i = 1, 2, \ldots, n$, after apparent computations we obtain:

$$\int_{\Delta t}^{T}\int_D \boldsymbol{U}_n^s(t-\Delta t)\frac{\partial \boldsymbol{\psi}_n}{\partial t}(t)\,dy\,dt \tag{4.64}$$

$$\begin{aligned}
= \quad & \int_0^T\int_D \Big\{-\frac{\partial h_n}{\partial t}\frac{\partial (y_2\boldsymbol{u}_n^s)}{\partial y_2}\cdot\boldsymbol{\psi}_n^s \\
& \quad +\left(h_n^s u_{1n}^s\left(\frac{\partial \boldsymbol{u}_n^s}{\partial y_1}-\frac{y_2}{h_n^s}\frac{\partial h_n^s}{\partial y_1}\frac{\partial \boldsymbol{u}_n^s}{\partial y_2}\right)+u_{2n}^s\frac{\partial \boldsymbol{u}_n^s}{\partial y_2}\right)\cdot\boldsymbol{\psi}_n^s \\
& \quad +\frac{h_n^s}{2}\,\boldsymbol{u}_n^s\cdot\boldsymbol{\psi}_n^s\,\mathrm{div}_{h_n^s}\boldsymbol{u}_n^s+\nu\frac{\partial \boldsymbol{\psi}_n^s}{\partial y_1}\cdot\left[h_n^s\left(\frac{\partial \boldsymbol{u}_n^s}{\partial y_1}-\frac{y_2}{h_n^s}\frac{\partial h_n^s}{\partial y_1}\frac{\partial \boldsymbol{u}_n^s}{\partial y_2}\right)\right] \\
& \quad +\nu\frac{\partial \boldsymbol{\psi}_n^s}{\partial y_2}\cdot\left[\frac{1}{h_n^s}\frac{\partial \boldsymbol{u}_n^s}{\partial y_2}-y_2\frac{\partial h_n^s}{\partial y_1}\left(\frac{\partial \boldsymbol{u}_n^s}{\partial y_1}-\frac{y_2}{h_n^s}\frac{\partial h_n^s}{\partial y_1}\frac{\partial \boldsymbol{u}_n^s}{\partial y_2}\right)\right] \\
& \quad -h_n^s\,q_n^s\,\mathrm{div}_{h_n^s}\boldsymbol{\psi}_n^s\Big\}\;dy\,dt \\
& +\int_0^T\int_0^1 \ell\left(q_{out}^{n,s}-\frac{1}{2}|u_{1n}^s|^2\right)\psi_{1n}^s\,(L,y_2,t)\,dy_2dt \\
& -\int_0^T\int_0^1 \ell\left(q_{in}^{n,s}-\frac{1}{2}|u_{1n}^s|^2\right)\psi_{1n}^s\,(0,y_2,t)\,dy_2dt \\
& +\int_0^T\int_0^L \Big\{q_w^{n,s}-\frac{1}{2}u_{2n}^s\left(u_{2n}^s-\frac{\partial h_n}{\partial t}\right) \\
& \quad +\kappa\left(u_{2n}^s-\lambda u_n^s-(1-\lambda)\frac{\partial h_n}{\partial t}\right)\Big\}\,\psi_{2n}^s\,(y_1,1,t)\,dy_1dt \\
& +\varepsilon\int_0^T\left\langle\frac{\partial Q_n}{\partial t},\phi_n^s\right\rangle dt \\
& +\varepsilon\int_0^T\int_D\Big\{-\frac{\partial h_n}{\partial t}\frac{\partial (y_2 q_n^s)}{\partial y_2}\phi_n^s+\varepsilon\frac{\partial \phi_n^s}{\partial y_1}\left[h_n^s\left(\frac{\partial q_n^s}{\partial y_1}-\frac{y_2}{h_n^s}\frac{\partial h_n^s}{\partial y_1}\frac{\partial q_n^s}{\partial y_2}\right)\right] \\
& \quad +\varepsilon\frac{\partial \phi_n^s}{\partial y_2}\left[\frac{1}{h_n^s}\frac{\partial q_n^s}{\partial y_2}-y_2\frac{\partial h_n^s}{\partial y_1}\left(\frac{\partial q_n^s}{\partial y_1}-\frac{y_2}{h_n^s}\frac{\partial h_n^s}{\partial y_1}\frac{\partial q_n^s}{\partial y_2}\right)\right] \\
& \quad +h_n^s\,\mathrm{div}_{h_n^s}\boldsymbol{u}_n^s\;\phi_n^s\Big\}\,dy\,dt \\
& \quad +\frac{\varepsilon}{2}\int_0^T\int_0^L\frac{\partial h_n}{\partial t}(y_1,t)\;q_n^s\phi_n^s\;(y_1,1,t)\,dy_1dt \\
& +\int_0^T\int_0^L\Big\{\frac{\partial u_n}{\partial t}\xi_n^s+c\,\frac{\partial u_n^s}{\partial y_1}\frac{\partial \xi_n^s}{\partial y_1}+a\frac{\partial}{\partial y_1}\int_0^t u_n^s(y_1,s)ds\frac{\partial \xi_n^s}{\partial y_1} \\
& \quad +b\int_0^t u_n^s(y_1,s)ds\;\xi_n^s \\
& \quad +\frac{\kappa}{E}\left(\lambda u_n^s+(1-\lambda)\frac{\partial h_n}{\partial t}-u_{2n}^s\right)\xi_n^s\Big\}\,(y_1,t)\,dy_1dt,
\end{aligned}$$

where $q_w^{n,s}(y_1,t)=g_w^i(y_1)=\frac{1}{\Delta t}\int_{(i-1)\Delta t}^{i\Delta t}q_w(y_1,s)ds,\ q_{in}^{n,s}(y_2,t)=q_{in}^i(y_2),\ q_{out}^{n,s}(y_3,t)=$

$q^i_{out}(y_2)$ for $t \in ((i-1)\Delta t, i\Delta t)$, see definitions in Section 4.1.

Remark 4.3. Passing to the limit for $n \to \infty$ yields for q_{in} and analogically for $q_w,\ q_{out}$,

$$q^{n,s}_{in} \to q_{in} \text{ in } L^2((0,1) \times (0,T))$$

because

$$\begin{aligned} & \int_0^T \int_0^1 |q^{n,s}_{in} - q_{in}|^2 dx_2\ dt \\ = & \sum_{i=1}^n \int_{(i-1)\Delta t}^{i\Delta t} \int_0^1 \frac{1}{\Delta t} \int_{(i-1)\Delta t}^{i\Delta t} |q_{in}(x_2,s) - q_{in}(x_2,t)|^2 ds\ dx_2\ dt \\ \le & \sum_{i=1}^n \int_{(i-1)\Delta t}^{i\Delta t} \int_0^1 \frac{1}{\Delta t} \int_{-\Delta t}^{\Delta t} |q_{in}(x_2,t+\tau) - q_{in}(x_2,t)|^2 d\tau\ dx_2\ dt \\ = & \frac{1}{\Delta t} \int_{-\Delta t}^{\Delta t} \int_0^T \int_0^1 |q_{in}(x_2,t+\tau) - q_{in}(x_2,t)|^2 dx_2\ dt\ d\tau \longrightarrow 0 \end{aligned}$$

if $\Delta t \to 0$, for 2-mean continuous function q_{in}, see [KJF77, Theorem 2.4.2, p.70]. For convergence of Steklov's averaging $[u]_\delta(t)\ \forall t \in (0, T-\delta)$, see e.g. [LSU68, Lemma 4.7, p.85]. $\diamond$

Passing to the limit $n \longrightarrow \infty$, we deduce that $(\boldsymbol{u}, q, u)$ satisfy (3.17) for test functions $(\boldsymbol{\psi}, \phi, \xi)$ with the stated regularity (4.63), where, however, the leading term

$$\int_0^T \left\langle \frac{\partial(h\boldsymbol{u})}{\partial t}, \boldsymbol{\psi} \right\rangle dt \quad \text{is replaced by} \quad -\int_0^T \int_D h\boldsymbol{u} \frac{\partial \boldsymbol{\psi}}{\partial t}\ dydt.$$

Due to the approximation argument we can conclude that (3.17) holds for every $(\boldsymbol{\psi}, \phi, \xi) \in L^2(0,T;V), \boldsymbol{\psi} \in L^4(0,T;L^4(D)^2)$.

It remains to show the existence of the time derivative of $h\boldsymbol{u}$. We follow Temam [Tem79]. Since $L^4(0,T;\boldsymbol{V}) \subset L^4(0,T;L^4(D)^2) \cap L^2(0,T;\boldsymbol{V})$, the equality (3.17) holds also for $(\boldsymbol{\psi}, 0, 0) \in L^4(0,T;V)$. Hence there exists

$$\mathcal{L} \in \left(L^4(0,T;\boldsymbol{V})\right)^* = L^{4/3}(0,T;\boldsymbol{V}^*)$$

such that

$$-\int_0^T \int_D h\boldsymbol{u} \frac{\partial \boldsymbol{\psi}}{\partial t}\ dydt = \int_0^T \langle \mathcal{L}(t), \boldsymbol{\psi}(t) \rangle_V\, dt \tag{4.65}$$

for any $\boldsymbol{\psi} \in L^4(0,T;\boldsymbol{V}) \cap H^{1,1}(0,T;L^2(D)^2)$, $\boldsymbol{\psi}(T) = 0$, where $\langle \cdot, \cdot \rangle_V$ denotes the pairing between $\boldsymbol{V}^*$ and $\boldsymbol{V}$. Putting

$$\int_0^T \langle h\boldsymbol{u}, \boldsymbol{\varpi} \rangle_V\, dt = \int_0^T \int_D h\boldsymbol{u} \cdot \boldsymbol{\varpi}\ dydt \qquad \text{for } \boldsymbol{\varpi} \in L^1(0,T;\boldsymbol{V}),$$

(4.65) easily yields for $\boldsymbol{\psi}(x,t) = \boldsymbol{w}(x)\xi(t)$

$$-\int_0^T \langle h\boldsymbol{u}(t), \boldsymbol{w}\rangle_V \, \xi'(t)\; dt = \int_0^T \langle \mathcal{L}(t), \boldsymbol{w}\rangle_V \, \xi(t)\; dt \qquad \forall \boldsymbol{w} \in \boldsymbol{V}$$

and for all $\xi \in C_0^\infty(0,T)$, i.e. $\mathcal{L}$ is the distributive derivate

$$\mathcal{L} \;=\; \frac{\partial(h\boldsymbol{u})}{\partial t} \;\in\; L^{4/3}(0,T;\boldsymbol{V}^*). \tag{4.66}$$

Moreover, it is not difficult to see that

$$-\int_0^T \int_D h\boldsymbol{u}\frac{\partial \boldsymbol{\psi}}{\partial t}\; dydt \;=\int_0^T \left(\langle \mathcal{F}(t), \boldsymbol{\psi}(t)\rangle_V \;+\; \int_D \mathcal{N}(t)\cdot \boldsymbol{\psi}(t)\; dy\right) dt$$

for all

$$\boldsymbol{\psi} \in X \equiv L^2(0,T;\boldsymbol{V}) \cap L^4(0,T;L^4(D)^2)$$

and for given $\mathcal{F} \in L^2(0,T;\boldsymbol{V}^*)$ and $\mathcal{N} \in L^{4/3}(0,T;L^{4/3}(D)^2)$. Therefore, (4.65) and (4.66) yield

$$\int_0^T \left\langle \frac{\partial(h\boldsymbol{u})}{\partial t}, \boldsymbol{\psi}\right\rangle_V \; dt \;=\int_0^T \left(\langle \mathcal{F}(t), \boldsymbol{\psi}(t)\rangle_V \;+\; \int_D \mathcal{N}(t)\cdot \boldsymbol{\psi}(t)\; dy\right) dt \tag{4.67}$$

for any $\boldsymbol{\psi} \in L^4(0,T;\boldsymbol{V})$.

As the right-hand side of (4.67) is well-defined for any $\boldsymbol{\psi} \in X$, we define

$$\int_0^T \left\langle \frac{\partial(h\boldsymbol{u})}{\partial t}, \boldsymbol{\psi}\right\rangle dt \;=\int_0^T \left(\langle \mathcal{F}(t), \boldsymbol{\psi}(t)\rangle_V \;+\; \int_D \mathcal{N}(t)\cdot \boldsymbol{\psi}(t)\; dy\right) dt \tag{4.68}$$

for any $\boldsymbol{\psi} \in X$. Consequently

$$\frac{\partial(h\boldsymbol{u})}{\partial t} \in X^* = L^2(0,T;\boldsymbol{V}^*) + L^{4/3}(0,T;L^{4/3}(D)^2), \tag{4.69}$$

see [GGZ74, Theorem 5.13, p. 25].

The solution $\boldsymbol{u}(x,t)$ does not have the time derivative in $L^1(0,T;L^2(D)^2)$. Nevertheless, it has the following property.

Lemma 4.4. *Assume that item 1 of Definition 3.1 is satisfied. Then for almost all t, the following formula holds*

$$\frac{1}{2}\int_D |\boldsymbol{u}|^2(t)h(t)\; dy + \frac{1}{2}\int_0^t \int_D |\boldsymbol{u}|^2 \,\frac{\partial h}{\partial t}\; dy\, ds = \int_0^t \left\langle \frac{\partial(\boldsymbol{u}h)}{\partial t}, \boldsymbol{u}\right\rangle ds \tag{4.70}$$

and the same holds for function q.

Proof of Lemma 4.4.
1. Following Alt's-Luckhaus' idea from [AL83, Lemma 1.5]: If $A, B \in R$ and $a, b \in R_+$ then

$$A^2a - B^2b = 2(Aa - Bb)A - AB(a-b) - (A\sqrt{a} - B\sqrt{b})^2 + AB(\sqrt{a} - \sqrt{b})^2$$

and

$$A^2a - B^2b = 2(Aa - Bb)B - AB(a-b) + (A\sqrt{a} - B\sqrt{b})^2 - AB(\sqrt{a} - \sqrt{b})^2.$$

Therefore we have for almost all $t > 0$ point-wise in D

$$\begin{aligned} &|\boldsymbol{u}|^2 h\ (t) - |\boldsymbol{u}|^2 h\ (t-\Delta t) \\ &\leq 2\left[\boldsymbol{u}h\ (t) - \boldsymbol{u}h\ (t-\Delta t)\right]\cdot \boldsymbol{u}(t) - \boldsymbol{u}(t)\cdot\boldsymbol{u}(t-\Delta t)\left[h(t) - h(t-\Delta t)\right] \\ &\quad + \boldsymbol{u}(t)\cdot\boldsymbol{u}(t-\Delta t)\left[\sqrt{h(t)} - \sqrt{h(t-\Delta t)}\right]^2 \end{aligned} \tag{4.71}$$

and

$$\begin{aligned} &|\boldsymbol{u}|^2 h\ (t) - |\boldsymbol{u}|^2 h\ (t-\Delta t) \\ &\geq 2\left[\boldsymbol{u}h\ (t) - \boldsymbol{u}h\ (t-\Delta t)\right]\cdot \boldsymbol{u}(t-\Delta t) - \boldsymbol{u}(t)\cdot\boldsymbol{u}(t-\Delta t)\left[h(t) - h(t-\Delta t)\right] \\ &\quad - \boldsymbol{u}(t)\cdot\boldsymbol{u}(t-\Delta t)\left[\sqrt{h(t)} - \sqrt{h(t-\Delta t)}\right]^2, \end{aligned} \tag{4.72}$$

where $\boldsymbol{u}(t) = \mathbf{0}$ for $-\Delta t \leq t \leq 0$.
2. We then integrate the above inequalities over $D \times (0, \tau)$ and divide by Δt. The first two terms on the right-hand side equal

$$\begin{aligned} &2\int_0^\tau\int_D \left[\frac{\boldsymbol{u}h\ (t) - \boldsymbol{u}h\ (t-\Delta t)}{\Delta t}\right]\cdot\boldsymbol{u}(t)\,dy\,dt \\ &-\int_0^\tau\int_D \boldsymbol{u}(t)\cdot\boldsymbol{u}(t-\Delta t)\left[\frac{h(t) - h(t-\Delta t)}{\Delta t}\right]\,dy\,dt \\ &= -2\int_0^\tau \left\langle \frac{\partial(\boldsymbol{u}h)}{\partial t}, [\boldsymbol{u}]_{\Delta t}\right\rangle dt + 2\int_D [\boldsymbol{u}]_{\Delta t}(\tau)\cdot\boldsymbol{u}(\tau)\,h(\tau)\,dy \\ &\quad + \int_0^\tau\int_D \boldsymbol{u}(t)\cdot\boldsymbol{u}(t-\Delta t)\left[\frac{h(t) - h(t-\Delta t)}{\Delta t}\right]\,dy\,dt. \end{aligned}$$

Since the left-hand side of (4.71) after the integration over $D \times (0, \tau)$ equals to

$\int_{\tau-\Delta t}^{\tau}\int_D |\boldsymbol{u}|^2 h(t) dy\, dt$, we arrive at

$$\begin{aligned}
&\int_D \left[|\boldsymbol{u}|^2 h\right]_{\Delta t}(\tau) \\
&\le -2\int_0^\tau \left\langle \frac{\partial(\boldsymbol{u}h)}{\partial t}, [\boldsymbol{u}]_{\Delta t}\right\rangle dt + 2\int_D [\boldsymbol{u}]_{\Delta t}(\tau)\cdot \boldsymbol{u}(\tau)\, h(\tau)\, dy \\
&\quad + \int_0^\tau \int_D \boldsymbol{u}(t)\cdot \boldsymbol{u}(t-\Delta t)\left[\frac{h(t)-h(t-\Delta t)}{\Delta t}\right] dy\, dt \\
&\quad - \int_0^\tau \int_D \boldsymbol{u}(t)\cdot \boldsymbol{u}(t-\Delta t)\left(\frac{\sqrt{h(t)}-\sqrt{h(t-\Delta t}}{\Delta t}\right)^2 \Delta t\, dy\, dt
\end{aligned}$$

where we denote Steklov's average

$$[\boldsymbol{u}]_{\Delta t}(t) \equiv \frac{1}{\Delta t}\int_{t-\Delta t}^{t} \boldsymbol{u}(s)\, ds.$$

We obtain the reverse inequality from (4.72) in a similar way.
3. Finally, let $\Delta t \longrightarrow 0$ in the above inequality. We obtain

$$\begin{aligned}
&\int_D |\boldsymbol{u}|^2(\tau) h(\tau)\, dy \\
&\le -2\int_0^\tau \left\langle \frac{\partial(\boldsymbol{u}h)}{\partial t}, \boldsymbol{u}\right\rangle dt + 2\int_D |\boldsymbol{u}|^2(\tau)\, h(\tau)\, dy + \int_0^\tau \int_D |\boldsymbol{u}|^2 \frac{\partial h}{\partial t}\, dy\, dt.
\end{aligned}$$

This and the reverse inequality imply (4.70). ■

4.4 Uniqueness and continuous dependence on data

The main goal of this section is to show a continuous dependence of solutions to the problem (3.1)–(3.12) on the data h, q_{in}, q_w and q_{out}.

Theorem 4.4 (Continuous dependence on data)**.** *Let* $(\boldsymbol{u}^1, q^1, u^1)$ *and* $(\boldsymbol{u}^2, q^2, u^2)$ *be weak solutions to the initial boundary value problem (3.1)–(3.12) in the sense of Definition 3.1 with given functions* h^1, q^1_{in}, q^1_w, q^1_{out} *and* h^2, q^2_{in}, q^2_w, q^2_{out}*, respectively, and suppose that*

$$0 < \alpha \le h^j(y_1, t) \le \alpha^{-1}, \quad \left|\frac{\partial h^j}{\partial y_1}(y_1,t)\right| + \left|\frac{\partial h^j}{\partial t}(y_1,t)\right| \le K \tag{4.73}$$

for given α, K *and for all* $(y_1, t) \in [0, L] \times [0, T]$, $j = 1, 2$. *Then for almost all* $t \in [0, T]$ *holds:*

$$
\begin{aligned}
&\int_D \left|h^1\boldsymbol{u}^1 - h^2\boldsymbol{u}^2\right|^2(t)\,dy + \nu\int_0^t\int_D \left|\nabla\left(h^1\boldsymbol{u}^1 - h^2\boldsymbol{u}^2\right)\right|^2 dy\,ds \\
&+\varepsilon\left(\int_D \left|h^1q^1 - h^2q^2\right|^2(t)\,dy + \int_0^t\int_D \left|\nabla\left(h^1q^1 - h^2q^2\right)\right|^2 dy\,ds\right) \\
&+\int_0^L \left|u^1 - u^2\right|^2(t)\,dy_1 + c\int_0^t\int_0^L \left|\frac{\partial(u^1-u^2)}{\partial y_1}\right|^2 dy_1\,ds \\
&+a\int_0^L\left(\int_0^t\left(\frac{\partial(u^1-u^2)}{\partial y_1}\right)ds\right)^2 dy_1 + b\int_0^L\left(\int_0^t (u^1-u^2)\,ds\right)^2 dy_1 \\
&\qquad\leq \|h^1 - h^2\|^2_{W^{1,\infty}((0,L)\times(0,t))}\,\omega(t) \\
&+C\int_0^t\left(\|q^1_{out} - q^2_{out}\|^2_{L^2(S_{out})} + \|q^1_{in} - q^2_{in}\|^2_{L^2(S_{in})} + \|q^1_w - q^2_w\|^2_{L^2(S_w)}\right)ds,
\end{aligned}
\tag{4.74}
$$

where $\omega(t) \downarrow 0$ *as* $t \to 0$, $i = 1, 2$ *and* $C > 0$.

Proof of Theorem 4.4.
1. First note that

$$
\boldsymbol{\zeta} = (\boldsymbol{\psi}, \phi, \xi) \equiv (h^1\boldsymbol{u}^1 - h^2\boldsymbol{u}^2, h^1q^1 - h^2q^2, E(u^1 - u^2)) \tag{4.75}
$$

is an admissible test function in the weak formulation of our problem for both $\boldsymbol{w}^1 = (\boldsymbol{u}^1, q^1, u^1)$ and $\boldsymbol{w}^2 = (\boldsymbol{u}^2, q^2, u^2)$, see (3.17). Therefore, we can subtract both identities. After tedious but straightforward manipulations and with the assistance of Lemma 4.4, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_D \left(|\boldsymbol{\psi}|^2+\varepsilon|\phi|^2\right)(t)\,dy+\frac{E}{2}\int_O^L |\xi|^2(t)\,dy_1\\
&+\int_0^t \left\{\nu a(\psi_1,\psi_1)+\nu a(\psi_2,\psi_2)+\varepsilon a(\phi,\phi)+Ec\|\xi\|^2_{H^1_0(0,L)}\right\}ds\\
&+\frac{Ea}{2}\int_0^L\left(\int_0^t\frac{\partial\xi}{\partial y^1}ds\right)^2 dy_1+\frac{Eb}{2}\int_0^L\left(\int_0^t \xi\,ds\right)^2 dy_1\\
=\;&\int_0^t \left(b^1(\boldsymbol{w}^1,\boldsymbol{w}^1,\boldsymbol{\zeta})-b^2(\boldsymbol{w}^2,\boldsymbol{w}^2,\boldsymbol{\zeta})\right)\,ds \qquad (4.76)\\
&+\int_0^t\int_0^L \kappa\,(\xi-\psi_2)\left(\lambda\xi+(1-\lambda)\left(\frac{\partial h^1}{\partial t}-\frac{\partial h^2}{\partial t}\right)-(u_2^1-u_2^2)\right)dy_1\;ds\\
&+\int_0^t\int_0^1\left((q^1_{out}-q^2_{out})\psi_1(L)-(q^1_{in}-q^2_{in})\psi_1(0)\right)\,dy_2\;ds\\
&+\int_0^t\int_0^L (q^1_w-q^2_w)\psi_2(1)dy_1ds\\
&-\int_0^t\int_D h^1q^1div_{h^1}\boldsymbol{\psi}-h^2q^2div_{h^2}\boldsymbol{\psi}-\left(h^1\phi\;div_{h^1}\boldsymbol{u}^1-h^2\phi\;div_{h^2}\boldsymbol{u}^2\right)dy\;ds\\
&+\int_0^t\left\{\frac{1}{2}\int_0^L\left[\frac{1}{h^1}\frac{\partial h^1}{\partial t}-\frac{1}{h^2}\frac{\partial h^2}{\partial t}\right]h^2(u_2^2\psi_2+q^2\phi)+\frac{1}{h^1}\frac{\partial h^1}{\partial t}\left((\psi_2)^2+\phi^2\right)\,dy_1\right.\\
&\qquad-\int_D\frac{1}{h^1}\frac{\partial h^1}{\partial t}\left(\frac{\partial\,(y_2\boldsymbol{\psi})}{\partial y_2}\cdot\boldsymbol{\psi}+\frac{\partial\,(y_2\phi)}{\partial y_2}\phi\right)\,dy\;ds\\
&\qquad\left.-\int_D\left[\frac{1}{h^1}\frac{\partial h^1}{\partial t}-\frac{1}{h^2}\frac{\partial h^2}{\partial t}\right]\left(\frac{\partial\left(y_2\boldsymbol{u}^2\right)}{\partial y_2}\cdot\boldsymbol{\psi}+\frac{\partial\left(y_2q^2\right)}{\partial y_2}\phi\right)h^2dy\;\right\}ds\\
&+\nu\left(R(\psi_1)+R(\psi_2)\right)+\;\varepsilon R(\phi),
\end{aligned}
$$

where we recall the notation from (4.4)

$$
\begin{aligned}
b^j(\boldsymbol{w},\boldsymbol{m},\boldsymbol{\zeta})=B^j(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\omega})+\int_D\frac{h^j}{2}\,\boldsymbol{z}\cdot\boldsymbol{\omega}\,\mathrm{div}_{h^j}\,\boldsymbol{u}\,dy\\
-\frac{\ell}{2}\int_0^1 u_1z_1\omega_1\,(L,y_2)\,dy_2+\frac{\ell}{2}\int_0^1 u_1z_1\omega_1\,(0,y_2)\,dy_2\\
-\frac{1}{2}\int_0^L u_2z_2\omega_2\,(y_1,1)\,dy_1, \qquad (4.77)
\end{aligned}
$$

for $\boldsymbol{w}=(\boldsymbol{u},\cdot,\cdot),\;\boldsymbol{m}=(\boldsymbol{z},\cdot,\cdot),\;\boldsymbol{\zeta}=(\boldsymbol{\omega},\cdot,\cdot)$,

$$
B^j(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\omega})\equiv\int_D\left(h^ju_1\left(\frac{\partial\boldsymbol{z}}{\partial y_1}-\frac{y_2}{h^j}\frac{\partial h^j}{\partial y_1}\frac{\partial\boldsymbol{z}}{\partial y_2}\right)+u_2\frac{\partial\boldsymbol{z}}{\partial y_2}\right)\cdot\boldsymbol{\omega}\,dy
$$

for $j = 1, 2$ and, in this notation

$$a(\phi, \phi) = \int_D \left(\frac{\partial \phi}{\partial y_1} - \frac{y_2}{h^2} \frac{\partial h^2}{\partial y_2} \frac{\partial \phi}{\partial y_2} \right)^2 + \frac{1}{(h^2)^2} \left(\frac{\partial \phi}{\partial y_2} \right)^2 \, dy$$

and

$$\begin{aligned} R(\phi) \quad = \quad & \int_0^t \int_D \frac{1}{2} \frac{\partial \phi^2}{\partial y_1} \frac{1}{h^2} \frac{\partial h^2}{\partial y_1} + \frac{1}{2} \frac{\partial \phi^2}{\partial y_2} \frac{y_2}{(h^2)^2} \left(\frac{\partial h^2}{\partial y_1} \right)^2 \\ & + \left[\frac{1}{h^1} \frac{\partial h^1}{\partial y_1} - \frac{1}{h^2} \frac{\partial h^2}{\partial y_1} \right] \left(\frac{\partial \phi}{\partial y_1} \frac{\partial (y_2 h^1 q^1)}{\partial y_2} + \frac{\partial \phi}{\partial y_2} \frac{y_2 \partial (h^1 q^1)}{\partial y_1} \right) \\ & - \left[\frac{1}{(h^1)^2} - \frac{1}{(h^2)^2} \right] \frac{\partial \phi}{\partial y_2} \frac{\partial (h^1 q^1)}{\partial y_2} \\ & + \left[\left(\frac{1}{h^1} \frac{\partial h^1}{\partial y_1} \right)^2 - \left(\frac{1}{h^2} \frac{\partial h^2}{\partial y_1} \right)^2 \right] \frac{\partial \phi}{\partial y_2} y_2 \left(h^1 q^1 - \frac{\partial (h^1 q^1)}{\partial y_2} \right) \, dy \, ds. \end{aligned} \tag{4.78}$$

2. To estimate the right-hand side of (4.76), we first compute $b^1(\boldsymbol{w}^1, \boldsymbol{w}^1, \boldsymbol{\zeta}) - b^2(\boldsymbol{w}^2, \boldsymbol{w}^2, \boldsymbol{\zeta})$. Let us recall (4.12), i.e.

$$b^j(\boldsymbol{w}, \boldsymbol{m}, \boldsymbol{\zeta}) = \frac{1}{2} B^j(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\omega}) - \frac{1}{2} B^j(\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{z}).$$

Thus it is not difficult to verify that

$$\begin{aligned} & b^1(\boldsymbol{w}^1, \boldsymbol{w}^1, \boldsymbol{\zeta}) - b^2(\boldsymbol{w}^2, \boldsymbol{w}^2, \boldsymbol{\zeta}) = \\ & \quad + \frac{1}{2} B^2(\boldsymbol{u}^1 - \boldsymbol{u}^2, \boldsymbol{u}^2, \boldsymbol{\psi}) - \frac{1}{2} B^2(\boldsymbol{u}^1 - \boldsymbol{u}^2, \boldsymbol{\psi}, \boldsymbol{u}^2) \\ & \quad + \frac{1}{2} B^1(\boldsymbol{u}^1, \boldsymbol{u}^1 - \boldsymbol{u}^2, \boldsymbol{\psi}) - \frac{1}{2} B^1(\boldsymbol{u}^1, \boldsymbol{\psi}, \boldsymbol{u}^1 - \boldsymbol{u}^2) \\ & \quad + \frac{1}{2} \int_D u_1^1 \left[h^1 - h^2 \right] \left(\frac{\partial \boldsymbol{u}^2}{\partial y_1} \cdot \boldsymbol{\psi} - \boldsymbol{u}^2 \cdot \frac{\partial \boldsymbol{\psi}}{\partial y_1} \right) \\ & \qquad - y_2 \, u_1^1 \left[\frac{\partial h^1}{\partial y_1} - \frac{\partial h^2}{\partial y_1} \right] \left(\frac{\partial \boldsymbol{u}^2}{\partial y_2} \cdot \boldsymbol{\psi} - \boldsymbol{u}^2 \cdot \frac{\partial \boldsymbol{\psi}}{\partial y_2} \right) \, dy. \end{aligned} \tag{4.79}$$

Recalling (3.4), after some manipulation we also obtain

$$\begin{aligned} & h^1 q^1 div_{h^1} \boldsymbol{\psi} - h^2 q^2 div_{h^2} \boldsymbol{\psi} - \left(h^1 \phi \; div_{h^1} \boldsymbol{u}^1 - h^2 \phi \; div_{h^2} \boldsymbol{u}^2 \right) = \\ & -y_2 \left[\frac{1}{h^1} \frac{\partial h^1}{\partial y_1} - \frac{1}{h^2} \frac{\partial h^2}{\partial y_1} \right] \left(h^1 q^1 \frac{\partial \psi_1}{\partial y^2} + h^2 \frac{\partial u_1^1}{\partial y^2} \phi \right) - \left[\frac{h^1 - h^2}{h^2} \right] \left(q^1 \frac{\partial \psi_2}{\partial y^2} + \frac{\partial u_2^1}{\partial y^2} \phi \right) \\ & - \left(\frac{\partial h^1}{\partial y_1} \frac{1}{h_2} \psi_1 + \frac{\partial h^1}{\partial y_1} u_1^1 \left[\frac{h^1 - h^2}{h^2} \right] + u_1^2 \left[\frac{\partial h^1}{\partial y_1} - \frac{\partial h^2}{\partial y_1} \right] \right) \phi. \end{aligned}$$

3. In the sequel, we attempt to estimate the right-hand side of (4.76) in order to get the following differential inequality

$$g(t) \leq \vartheta(t) + \int_0^t r(s) g(s) \, ds \qquad \forall t \in [0, T] \tag{4.80}$$

for

$$g(t) \equiv \int_D \frac{1}{2}\left(|\boldsymbol{\psi}|^2 + \varepsilon|\phi|^2\right)(t)\,dy + \int_0^L \frac{E}{2}|\xi|^2(t)\,dy_1$$

and some continuous function ϑ and integrable function r on $[0,T]$, $\vartheta, r \geq 0$. Gronwall's Lemma (see e.g. [Fei93, Lemma 8.2.29]) then yields

$$g(t) \leq \vartheta(t) + \int_0^t \vartheta(s)r(s)e^{\int_s^t r(\tau)d\tau}ds, \qquad t \in [0,T]. \tag{4.81}$$

4. Let us begin with the first term on the right-hand side of (4.79), i.e.

$$B^2(\boldsymbol{u}^1 - \boldsymbol{u}^2, \boldsymbol{u}^2, \boldsymbol{\psi}) = \int_D \left(h^2(u_1^1 - u_1^2)\left(\frac{\partial \boldsymbol{u}_1^2}{\partial y_1} - \frac{y_2}{h^2}\frac{\partial h^2}{\partial y_1}\frac{\boldsymbol{u}^2}{\partial y_2}\right) + (u_2^1 - u_2^2)\frac{\partial \boldsymbol{u}^2}{\partial y_2}\right)\cdot \boldsymbol{\psi}\,dy$$

and focus on its first term

$$\begin{aligned}
&\int_D h^2(u_1^1 - u_1^2)\frac{\partial \boldsymbol{u}^2}{\partial y_1}\cdot\boldsymbol{\psi}\,dy \\
&\quad = \int_D \frac{h^2}{h^1}\left((h^1 - h^2)u_1^1 + \psi_1\right)\frac{\partial \boldsymbol{u}^2}{\partial y_1}\cdot\boldsymbol{\psi}\,dy \\
&\quad \leq \left|\frac{h^2}{h^1}\right|_\infty \|\nabla \boldsymbol{u}^2\|_{L^2}\|\boldsymbol{\psi}\|_{L^4}\left(\left|h^1 - h^2\right|_\infty \|\boldsymbol{u}^1\|_{L^4} + \|\boldsymbol{\psi}\|_{L^4}\right) \\
&\quad \leq C_1\left|h^2 - h^1\right|_\infty^2 \|\nabla \boldsymbol{u}^2\|_{L^2}^2 + \mu_1\|\nabla\boldsymbol{\psi}\|_{L^2}^2 \\
&\qquad\qquad + C_2(\mu_1)\|\boldsymbol{\psi}\|_{L^2}^2\left(\|\nabla \boldsymbol{u}^1\|_{L^2}^2 + \|\nabla \boldsymbol{u}^2\|_{L^2}^2\right)
\end{aligned}$$

for $0 < \mu_1 < 1$, where μ is to be determined later. Here Proposition 3.1 has been applied. We omit similar computations which estimate the remaining terms of the right-hand side of (4.76) and note that

$$a(\phi,\phi) \geq \frac{\alpha^2}{2+K^2}\int_D |\nabla\phi|^2\,dy \qquad \forall \phi \in H^1(D),$$

see Lemma 4.2. Hence, we arrive at

$$\begin{aligned}
&g(t) + \left(\frac{\alpha^2}{2+K^2} - \mu_1\right)\nu\int_0^t \|\nabla\boldsymbol{\psi}\|_{L^2}^2 ds + \left(\frac{\alpha^2}{2+K^2} - \mu_2\right)\varepsilon\int_0^t \|\nabla\phi\|_{L^2}^2 ds \\
&+\frac{a}{2}\int_0^L\left(\int_0^t \frac{\partial\xi}{\partial y^1}ds\right)^2 dy_1 + \frac{b}{2}\int_0^L\left(\int_0^t \xi\,ds\right)^2 dy_1 \\
&+\kappa\int_0^t\left(\lambda\|\xi\|_{L^2(0,L)}^2 + \alpha\|\psi_2\|_{L^2(0,L)}^2\right)ds + c\int_0^t \|\nabla\xi\|_{L^2(0,L)}^2 ds \\
&\leq\ \vartheta(t) + \int_0^t r(s)g(s)\,ds,
\end{aligned} \tag{4.82}$$

where

$$\vartheta(t) \equiv C_3 \, \|h^1 - h^2\|^2_{W^{1,\infty}([0,L]\times[0,t])} \int_0^t \left(\|\nabla \boldsymbol{u}^2\|^2_{L^2} + \varepsilon \|\nabla q^2\|^2_{L^2} + 1\right) \, ds$$
$$+ C_4 \int_0^t \left(\|q^1_{out} - q^2_{out}\|^2_{L^2(S_{out})} + \|q^1_{in} - q^2_{in}\|^2_{L^2(S_{in})} + \|q^1_w - q^2_w\|^2_{L^2(S_w)} \right) \, ds$$

and

$$r(t) \equiv C_5(\kappa, \alpha, K, \mu_1, \mu_2)\Big(\|\nabla \boldsymbol{u}^1\|^2_{L^2}(t) + \|\nabla \boldsymbol{u}^2\|^2_{L^2}(t) + 1\Big),$$

$C_3 = C_3(\alpha, K, \kappa)$. Note that $\vartheta(t)$ is non-decreasing.
5. After choosing $\mu_1 = \mu_2 = \alpha^2/(4 + 2K^2)$ and omitting the positive terms on the left-hand side of (4.82) except of $g(t)$ and applying Gronwall's lemma (4.81), we can estimate

$$g(t) \leq \vartheta(t) \left[1 + \int_0^T r(\tau) \, d\tau e^{\int_0^T r(\tau) \, d\tau}\right] \equiv \vartheta(t) C_6. \tag{4.83}$$

Finally, we estimate $g(s)$ at the right-hand side of (4.82) with the assistance of (4.83). Thus,

$$\vartheta(t) + \int_0^t r(s) g(s) \, ds \leq \vartheta(t) \left[1 + C_6 \int_0^T r(\tau) \, d\tau\right] \equiv \vartheta(t) C_7. \tag{4.84}$$

The required estimate (4.74) follows easily if we look at (4.82), (4.84) for $\vartheta(t)$. Note that

$$\omega(t) \equiv C_3 C_7 C_8 \int_0^t \left(\|\nabla \boldsymbol{u}^2\|^2_{L^2} + \varepsilon \|\nabla q^2\|^2_{L^2} + 1\right) \, ds,$$

$C \equiv C_4 C_7 C_8$ and $C_8 \equiv 1/\min\left\{\frac{E}{2}, \frac{1}{2}, \frac{\alpha^2}{2(2+K^2)}\right\}$ in (4.74). ■

Remark 4.4. Note again that the test function used in the previous proof, $\boldsymbol{\psi} = h^1 \boldsymbol{u}^1 - h^2 \boldsymbol{u}^2$ does not fulfill the divergence-free condition, i.e. even if $\mathrm{div}_{h^i} \, \boldsymbol{u}^i = 0, \ i = 1, 2$ were found, this does not yet mean that

$$\mathrm{div}_{h^i} \, \boldsymbol{\psi} = \frac{\partial \psi_1}{\partial y_1} - \frac{y_2}{h^i} \frac{\partial h^i}{\partial y_1} \frac{\partial \psi_1}{\partial y_2} + \frac{1}{h^i} \frac{\partial \psi_2}{\partial y_2} = 0, \ i = 1, 2$$

since $h^1 \neq h^2$ in general. This is the second reason for the regularisation of the incompressibility condition (1.1) of original mathematical model, see (2.26). ◇

4.5 Problem with ε=0

The final goal of this section is to let

$$\varepsilon \longrightarrow 0$$

and to prove the existence of the solution to the problem (1.1)–(1.10) introduced in Chapter 1 which is a κ-approximation of the original Quarteroni's model (2.9)–(2.18).

We denote the solution to our regularised problem dependent on ε (whose existence and uniqueness was investigated in previous sections of this chapter) as $(\boldsymbol{u}_\varepsilon, q_\varepsilon, u_\varepsilon)$. We use the same techniques as before and show that subsequences $\boldsymbol{u}_\varepsilon$, u_ε converge strongly to a weak solution $\boldsymbol{u}$, u. Since we let ε go to zero, our main task is to derive a priori estimates which do not depend on ε.

1. We first obtain an a priori estimate by testing $(\boldsymbol{u}_\varepsilon, q_\varepsilon, Eu_\varepsilon)$. After putting $(\boldsymbol{\psi}, \phi, \xi) = (\boldsymbol{u}_\varepsilon, q_\varepsilon, Eu_\varepsilon)$ in (3.17) with T replaced by t and application of Lemma 4.4 and Gronwall's lemma, we obtain the a priori estimate

$$\begin{aligned}
&\int_D h(t)\left(|\boldsymbol{u}_\varepsilon|^2 + \varepsilon|q_\varepsilon|^2\right)(t)\,dy + \frac{\lambda E}{2}\int_0^L |u_\varepsilon(t)|^2\,dy_1 \\
&+\frac{\alpha}{2+K^2}\int_0^t\int_D \left(\nu\,|\nabla \boldsymbol{u}_\varepsilon|^2 + \varepsilon\,|\nabla q_\varepsilon|^2\right) dy\,ds \\
&+2\kappa\int_0^t\int_0^L \left|u_{2\varepsilon} - \left(\lambda u_\varepsilon + (1-\lambda)\frac{\partial h}{\partial t}\right)\right|^2 dy_1\,ds \\
&+\frac{\lambda E}{2}\left[c\int_0^t\int_0^L \left|\frac{\partial u_\varepsilon}{\partial y_1}\right|^2 dy_1\,ds + a\int_0^L\left|\int_0^t \frac{\partial u_\varepsilon}{\partial y_1}\,ds\right|^2 dy_1 + b\int_0^L\left|\int_0^t u_\varepsilon\,ds\right|^2 dy_1\right] \\
&\le\ \mathcal{R}(t)
\end{aligned} \tag{4.85}$$

for a.e. $t \in (0,T)$, where

$$\begin{aligned}
\mathcal{R}(t) \equiv &\left\{\frac{2E(1-\lambda)^2}{\lambda}\max_{0\le\tau\le t}\left[\int_0^L\left(\left|\frac{\partial h}{\partial t}\right|^2 + Ea\left|\frac{\partial h}{\partial y_1}\right|^2 + Eb\,|h|^2\right)(\tau)dy_1\right]\right. \\
&+\frac{2+K^2}{4\alpha}\int_0^t\left(\|q_{out}\|^2_{L^2(S_{out})} + \|q_{in}\|^2_{L^2(S_{in})} + \|q_w\|^2_{L^2(S_w)}\right)ds \\
&\left.+\frac{4(1-\lambda)^2E}{\lambda c}\int_0^t\int_0^L\left(\left|\frac{\partial^2 h}{\partial y_1\partial t}\right|^2 + \left|\frac{\partial^2 h}{\partial t^2}\right|^2 + a\left|\frac{\partial h}{\partial y_1}\right|^2 + b\,|h|^2\right)dy_1 ds\right\} e^{\frac{M}{\alpha}t}
\end{aligned}$$

which proves that subsequence $(\boldsymbol{u}_\varepsilon, \sqrt{\varepsilon}q_\varepsilon, u_\varepsilon)$ has a weak limit $(\boldsymbol{u}, \vartheta, u)$ in $L^2(0,T;V)$. We further show that

$$\mathrm{div}_h\, \boldsymbol{u} = 0 \qquad \text{a.e. on} \quad D\times(0,T). \tag{4.86}$$

To show (4.86), insert $(\boldsymbol{\psi}, \phi, \xi) = (\mathbf{0}, \phi, 0)$ for sufficiently smooth ϕ into (3.17), which yields

$$\begin{aligned}
&\int_0^T \int_D h\,\phi\, \mathrm{div}_h \boldsymbol{u}_\varepsilon \, dy\, dt && (4.87)\\
&= \varepsilon \int_0^T \int_D \left(h q_\varepsilon \frac{\partial \phi}{\partial t} + \frac{\partial h}{\partial t} \frac{\partial (y_2 q_\varepsilon)}{\partial y_2} \phi - \frac{\partial \phi}{\partial y_1} \left[h \left(\frac{\partial q_\varepsilon}{\partial y_1} - \frac{y_2}{h} \frac{\partial h}{\partial y_1} \frac{\partial q_\varepsilon}{\partial y_2} \right) \right] \right.\\
&\qquad \left. - \frac{\partial \phi}{\partial y_2} \left[\frac{1}{h} \frac{\partial q_\varepsilon}{\partial y_2} - y_2 \frac{\partial h}{\partial y_1} \left(\frac{\partial q_\varepsilon}{\partial y_1} - \frac{y_2}{h} \frac{\partial h}{\partial y_1} \frac{\partial q_\varepsilon}{\partial y_2} \right) \right] \right) dy\, dt\\
&- \frac{\varepsilon}{2} \int_0^T \int_0^L \frac{\partial h}{\partial t}(y_1, t)\; q_\varepsilon \phi\; (y_1, 1, t)\, dy_1 dt.
\end{aligned}$$

Due to the estimate (4.85) concerning q_ε, the right-hand side of (4.87) tends to zero as $\varepsilon \longrightarrow 0$ and we arrive at (4.86).
2. Moreover, using (4.85) we show the a priori estimate for time derivative of u_ε by testing (3.17) with $(0, 0, E\frac{\partial u}{\partial t})$.

$$\frac{E}{2} \int_0^L \int_0^t \left| \frac{\partial u}{\partial t} \right|^2 + \frac{cE}{4} \int_0^L \left| \frac{\partial u}{\partial y_1} \right|^2 \leq C(\kappa) \mathcal{R}(t) \tag{4.88}$$

This proves that $\{u_\varepsilon\}$ is bounded in $H^1((0, L) \times (0, T))$. It implies that this sequence converges strongly in $L^p((0, L) \times (0, T))$ for any $p > 0$ (see Proposition 3.1).
3. Then we derive the third a priori estimate by multiplying (3.17) with time differences

$$\left(\partial_t^\tau (h \boldsymbol{u}_\varepsilon) \;,\; \partial_t^\tau (h q_\varepsilon) \;,\; \partial_t^\tau \left(\lambda h u_\varepsilon + (1 - \lambda) h \frac{\partial h}{\partial t} \right) \right),$$

where we denote $\partial_t^\tau f \equiv f(t + \tau) - f(t)$ for $\tau > 0$. This results in the compactness of the functions $(\boldsymbol{u}_\varepsilon, \sqrt{\varepsilon} q_\varepsilon, u_\varepsilon)$ in L^2. To prove this compactness, we apply a technique similar to the technique used in Section 4.2 in the proof of the third a priori estimate (Theorem 4.3) and show that

$$\begin{aligned}
&\int_0^{T-\tau} \int_D \left(|(h\boldsymbol{u}_\varepsilon)(t+\tau) - (h\boldsymbol{u}_\varepsilon)(t)|^2 + \varepsilon \, |(hq_\varepsilon)(t+\tau) - (hq_\varepsilon)(t)|^2 \right) \, dy\, dt\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.89)\\
&+ \int_0^{T-\tau} \int_0^L |u_\varepsilon(t+\tau) - u_\varepsilon(t)|^2 \, dy_1 \, dt \;\; \leq \;\; C\,\tau
\end{aligned}$$

for a positive constant C independent on ε and τ. (4.89) can be obtained in the following way. Using $\chi^\delta_{(t,t+\tau)}\,(\boldsymbol{w}, p, v)$, $(\boldsymbol{w}, p, v) \in V$ as a test function in (3.17), where $\chi^\delta_{(t,t+\tau)}$ is a smooth approximation of the characteristic function of the interval $(t, t + \tau)$ yields after letting $\delta \to 0$:

$$\int_D (\partial_t^\tau (h \boldsymbol{u}_\varepsilon) \cdot \boldsymbol{w} \;+\; \varepsilon \partial_t^\tau (h q_\varepsilon) p)\, dy + \int_0^L \partial_t^\tau u_\varepsilon v \, dy_1 \;=\; \int_t^{t+\tau} [\,\ldots\,]\; ds \tag{4.90}$$

for a.e. $t \in (0, T-\tau)$. Now we put

$$\boldsymbol{w} = \partial_t^\tau(h\boldsymbol{u}_\varepsilon),\ p = \partial_t^\tau(hq_\varepsilon),\ v = \partial_t^\tau\left(\lambda h u_\varepsilon + (1-\lambda)h\frac{\partial h}{\partial t}\right)$$

and integrate (4.90) with respect to t over $(0, T-\tau)$. We arrive at

$$\begin{aligned}
&\int_0^{T-\tau}\int_D \left(|\partial_t^\tau(h\boldsymbol{u}_\varepsilon|^2 \ +\ \varepsilon\,|\partial_t^\tau(hq_\varepsilon)|^2\right) dy\ dt + \lambda\int_0^{T-\tau}\int_0^L h\,|\partial_t^\tau u_\varepsilon|^2\,dy_1 dt \\
&= -\int_0^{T-\tau}\int_0^L \left[\lambda(\partial_t^\tau u_\varepsilon)u_\varepsilon(t+\tau)\partial_t^\tau h + (1-\lambda)(\partial_t^\tau u_\varepsilon)\partial_t^\tau\left(h\frac{\partial h}{\partial t}\right)\right] dy_1 dt \\
&\quad + \int_0^{T-\tau}\int_t^{t+\tau}\left[\int_D \ldots dy \ +\ \int_0^L \ldots dy_1 \ +\ \int_0^1 \ldots dy_2\right] ds\ dt \ \le\ c(\kappa)\,\tau
\end{aligned}$$

We do not write the full form of the right-hand side terms, since they are analogous to the right-hand side terms in (4.29).

Due to the estimates (4.85), (4.89) and the compactness arguments of [AL83, Lemma 1.9], we can extract a subsequence of $\{\boldsymbol{u}_\varepsilon\}$, $\{u_\varepsilon\}$ which we again denote for simplicity as $\{\boldsymbol{u}_\varepsilon\}$, $\{u_\varepsilon\}$, such that

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \ \text{ in } \ L^1(D\times(0,T))^2, \quad u_\varepsilon \to u \ \text{ in } \ L^1((0,L)\times(0,T))$$

for $\varepsilon \to 0$. As $\boldsymbol{u}, \{\boldsymbol{u}_\varepsilon\}$ are bounded in $L^4(D\times(0,T))^2$ (see 3.1) and $u, \{u_\varepsilon\}$ are bounded in $L^p((0,L)\times(0,T))$ for any $p > 0$ (see item 2 above), then

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \ \text{ in } \ L^p(D\times(0,T))^2, \quad u_\varepsilon \to u \ \text{ in } \ L^q((0,L)\times(0,T)) \tag{4.91}$$

(strongly) for any $1 \le p < 4$ and $1 \le q < 6$ as $\varepsilon \to 0$.

If $\boldsymbol{\psi}$ is continuously differentiable, such that $\boldsymbol{\psi}(T) = 0$ and $\mathrm{div}_h\boldsymbol{\psi} = 0$, then we can pass to the limit in (3.17) using the weak convergence results

$$(\boldsymbol{u}_\varepsilon, \varepsilon q_\varepsilon, u_\varepsilon) \ \rightharpoonup\ (\boldsymbol{u}, 0, u)$$

in $L^2(0,T;V)$ and the strong convergence (4.91). We obtain

$$\begin{aligned}
&\int_0^T \int_D h\boldsymbol{u}\cdot\frac{\partial\boldsymbol{\psi}}{\partial t}dydt \\
&= \int_0^T \int_D \Bigg(-\frac{\partial h}{\partial t}\frac{\partial(y_2\boldsymbol{u})}{\partial y_2}\cdot\boldsymbol{\psi} + \left(hu_1\left(\frac{\partial\boldsymbol{u}}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial\boldsymbol{u}}{\partial y_2}\right) + u_2\frac{\partial\boldsymbol{u}}{\partial y_2}\right)\cdot\boldsymbol{\psi} \\
&\qquad + \frac{\partial\boldsymbol{\psi}}{\partial y_1}\cdot\left[\nu h\left(\frac{\partial\boldsymbol{u}}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial\boldsymbol{u}}{\partial y_2}\right)\right] \\
&\qquad + \frac{\partial\psi}{\partial y_2}\cdot\left[\frac{\nu}{h}\frac{\partial\boldsymbol{u}}{\partial y_2} - \nu y_2\frac{\partial h}{\partial y_1}\left(\frac{\partial\boldsymbol{u}}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial\boldsymbol{u}}{\partial y_2}\right)\right]\Bigg)\,dy\,dt \\
&+ \int_0^T\int_0^1 \ell\left(q_{out} - \frac{1}{2}|u_1|^2\right)\psi_1\ (L,y_2,t)\,dy_2dt \\
&- \int_0^T\int_0^1 \ell\left(q_{in} - \frac{1}{2}|u_1|^2\right)\psi_1\ (0,y_2,t)\,dy_2dt \\
&+ \int_0^T\int_0^L \left(q_w - \frac{1}{2}u_2\left(u_2 - \frac{\partial h}{\partial t}\right)\right)\psi_2\ (y_1,1,t)\,dy_1dt \\
&\kappa\int_0^T\int_0^L \left(u_2 - \lambda u - (1-\lambda)\frac{\partial h}{\partial t}\right)\psi_2\ (y_1,1,t)\,dy_1dt \\
&+ \int_0^T\int_0^L \Bigg(-u\frac{\partial\xi}{\partial t} + c\frac{\partial u}{\partial y_1}\frac{\partial\xi}{\partial y_1} + a\frac{\partial}{\partial y_1}\int_0^t u(y_1,s)ds\frac{\partial\xi}{\partial y_1} \\
&\qquad + b\int_0^t u(y_1,s)ds\ \xi + \frac{\kappa}{E}\left(\lambda u + (1-\lambda)\frac{\partial h}{\partial t} - u_2\right)\xi\Bigg)(y_1,t)\,dy_1dt.
\end{aligned} \tag{4.92}$$

Due to the approximation arguments, we can conclude that (4.92) holds for every $(\boldsymbol{\psi},\xi) \in L^2(0,T;\boldsymbol{V}\times H_0^1(0,L)), \boldsymbol{\psi} \in L^4(0,T;L^4(D)^2)$ and $\mathrm{div}_h\boldsymbol{\psi} = 0$. We have thus proved, analogously as in Section 4.3, the weak solution $(\boldsymbol{u},u)$ to our problem in the sense of (4.92), i.e. with the use of divergence-free test functions. Unfortunately, we can not prove the existence of distributive time derivative for velocity. The difficulty lies in the incompressibility condition $\mathrm{div}_h\ \boldsymbol{\psi} = 0$ for test functions.

We now turn back to the problem in the time-dependent domain $\Omega(h)$. We therefore define

$$\boldsymbol{v}(x_1,x_2,t) \stackrel{\text{def}}{=} \boldsymbol{u}\left(x_1, \frac{x_2}{h(x_1,t)}, t\right) \tag{4.93}$$

for $(x,t) \in \Omega(h)$ and

$$\eta(x_1,t) \stackrel{\text{def}}{=} \int_0^t u(x_1,s)\,ds \tag{4.94}$$

for $0 < x_1 < L$, $0 < t < T$. We can now rewrite the integral identity (4.92) as

$$\begin{aligned}
&\int_{\Omega(h)} \left\{ -\boldsymbol{v} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + \mu \nabla \boldsymbol{v} \cdot \nabla \boldsymbol{\varphi} + \sum_{i,j=1}^{2} v_i \frac{\partial v_j}{\partial x_i} \varphi_j \right\} dxdt \\
&+ \int_0^T \int_0^\ell \left(P_{out} - \frac{1}{2} |v_1|^2 \right) \varphi_1 \ (L, x_2, t) \, dy_2 dt \\
&- \int_0^T \int_0^\ell \left(P_{in} - \frac{1}{2} |v_1|^2 \right) \varphi_1 \ (0, x_2, t) \, dy_2 dt \\
&+ \int_0^T \int_0^L \left(P_w - \frac{1}{2} v_2 \left(v_2 - \frac{\partial h}{\partial t} \right) \right) \varphi_2 \ (x_1, h(x_1, t), t) \, dx_1 dt \qquad (4.95) \\
&+ \kappa \int_0^T \int_0^L \left(v_2 - \lambda \frac{\partial \eta}{\partial t} - (1-\lambda) \frac{\partial h}{\partial t} \right) \varphi_2 \ (x_1, 1, t) \, dx_1 dt \\
&+ \int_0^T \int_0^L \left(-\frac{\partial \eta}{\partial t} \frac{\partial \xi}{\partial t} + c \, \frac{\partial^2 \eta}{\partial x_1 \partial t} \frac{\partial \xi}{\partial x_1} + a \frac{\partial \eta}{\partial x_1} \frac{\partial \xi}{\partial y_1} \right. \\
&\qquad \left. + b \eta \, \xi + \frac{\kappa}{E} \left(\lambda \frac{\partial \eta}{\partial t} + (1-\lambda) \frac{\partial h}{\partial t} - u_2 \right) \xi \right) (x_1, t) \, dx_1 dt = 0
\end{aligned}$$

for

$$\boldsymbol{\varphi}(x_1, x_2, t) \stackrel{\text{def}}{=} \boldsymbol{\psi} \left(x_1, \frac{x_2}{h(x_1, t)}, t \right). \qquad (4.96)$$

Consequently, we have proved the following statement.

Theorem 4.5 (Existence of the weak solution)**.** *Let assumptions (3.15) hold. Then there exist a weak solution* $(\boldsymbol{v},\ \eta)$ *to the problem (1.1)–(1.10) in the sense of integral identity (4.95) for any* $(\boldsymbol{\varphi}, \xi)$*.*

Remark 4.5 (ε-regularisation of the continuity equation). Note that we have proved the existence of the weak solution to our κ-approximation of the original problem (1.1)–(1.10). Our κ-approximation was also dependent on an additional parameter ε. We can now get rid of the parameter ε, i.e. we can turn back to the original continuity equation instead of the regularisation

$$\varepsilon \left(\frac{\partial p}{\partial t} - \Delta p \right) + \operatorname{div} \boldsymbol{v}_\varepsilon = 0$$

for the problem defined in time-dependent domain. ◇

Chapter 5

Numerical approximation

We designed numerical methods for solving both the problems, the original Quarteroni's model with an unknown deformation η (2.1)–(2.8) and the approximated problem with a given domain deformation h (1.1)–(1.10). We implemented these methods in the UG software toolbox.

This chapter is divided into two sections. Section 5.1, is devoted to the numerical solution to the problem (2.1)–(2.8). Section 5.2 describes a numerical solution of our approximated problem (1.1)–(1.10). From the numerical point of view, the main difference between these two problems is the right-hand side for the domain deformation equation. The method of decoupling the flow and domain geometry (the global method) is common for both the problems. This method assumes an a priori knowledge of the domain deformation. More precisely, the fluid and the domain deformation are decoupled independently on the time discretisation. The fluid flow is computed in a deforming domain with an explicitly given deformation $h = \ell + \eta^{(k)}$, where $\eta^{(k)}$ is given ($\ell = R_0$ is the reference radius). Parallel to this non-stationary process, $\eta^{(k+1)}(x_1, t)$ is sequentially computed for all $t \in (0, T)$. We repeat this 'domain' iteration, where we point out that in k-th iteration, $h(x_1, t) \equiv \ell + \eta^{(k-1)}(x_1, t),\ k = 1, \dots K$. A description of this algorithm can be found in Subsections 5.1.1 and 5.2.1.

The above algorithm involves solving non-stationary Navier-Stokes NS equations in a domain, whose geometry changes in time. The Navier-Stokes problem class for moving domain in UG Software toolbox was implemented by P.J. Broser from the University of Heidelberg. The details concerning the implementation of moving and smoothing the grid can be found e.g. in his diploma thesis [Bro02]. The finite volume method (FVM) is used for solving the velocity field. This method involves a stabilisation of the continuity equation and a linearisation of the momentum equation. The implementation of this time and space discretisation method into UG software toolbox is described briefly in [Bro02] and in more detail in [Näg03]. The finite volume method for the Navier-Stokes problem can be found e.g. in [QV97]. We use the finite difference method (FDM) to solve the deformation of the elastic wall η. This method and other details concerning the numerical discretisation* for Quarteroni's model are described in Subsection 5.1.2. The methods of discretisation for our model approximating the fluid-domain interface condition are quite similar to the method used for Quarteroni's original problem. However, there are some differences, which can be found in Subsection 5.2.2.

In Subsections 5.1.3 and 5.2.3, we present numerical experiments with both the problems. Some observations and conclusions concerning the numerical part of this work can be found in Section 5.3.

5.1 Original Quarteroni's problem

The problem (2.1)–(2.8) is a coupled fluid–structure problem, where the structure problem is described by (2.10)–(2.11), e.g. by the generalised string model for the domain deformation

$$\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x^2} + b\eta - c\frac{\partial^3\eta}{\partial t\partial x^2} = H(x,t), \tag{5.1}$$

where

$$H(x,t) = \frac{1}{\rho\rho_w h}\left(p - P_w - \mu\left((\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T)\cdot\boldsymbol{n}\right)\cdot\boldsymbol{e}_2\right), \boldsymbol{n} = \left(-\frac{\partial\eta}{\partial x_1}, 1\right).$$

Under the assumption that the stress tensor of the fluid $\boldsymbol{T}_f$ is isotropic, which means $\boldsymbol{T} = -p\mathbf{I}$, we obtain the following forcing term on the right-hand side:

$$H(x,t) = \frac{(p - P_w)}{\rho\rho_w h}.$$

This assumptions was also made in [FGNQ00].

5.1.1 Decoupling the fluid flow and the domain geometry

As we have already mentioned, the coupling between the fluid and domain is twofold. First, the stress tensor of the fluid influences the domain deformation, since it appears on the right-hand side of the structure equation. On the other hand, the Dirichlet boundary condition (2.12) on Γ_t^{wall} for the velocity is related to the domain deformation η. Under the assumption that the wall deformation is small, i.e. $\boldsymbol{n} \approx \boldsymbol{e}_2$, the Dirichlet condition reads:

$$\frac{\partial\eta}{\partial t} = v_2 \equiv v_2^{grid},$$

*By the discretisation of the deformation equation, we denote the longitudinal variable $x \equiv x_1$ in order to keep the notation in this chapter short.

where $\boldsymbol{v}^{grid}$ is the velocity of the mesh movement related to smoothing the grid after moving its boundary. In our case

$$\boldsymbol{v}^{grid} = \left(0, \frac{\partial \eta^0(x,t)}{\partial t}\right) \text{ on } \Gamma_t^{wall},$$

where $\eta^{(0)}(x,t)$ is some approximation of the domain deformation. For the horizontal velocity of the fluid we use the homogeneous Dirichlet boundary condition, as it is designed in (2.20). We use the time discretisation, and decouple the fluid equation from the equation for η in each time step, since they are coupled through (2.10)–(2.11) and (2.12). Moreover, we iterate with respect to the domain deformation. We describe this iterative algorithm in the following subsection.

Global iterative method with respect to the domain $\Omega_t^{(k)}$

Let K denote the (given) number of domain iterations, let T denote the (given) end of time interval, let $k = 0$.

0. Let
   $\Omega_t^{(k)} \equiv \{(x, x_2(t)),\ |x_2(t)| < R_0 + \eta^{(k)}(x,t),\ x \in (0,L)\}$, $t \in (0,T)$, $(\eta^{(0)} \equiv 0)$ be the given computational domain.
   Let $n = 0$, let the initial condition be
   $\boldsymbol{v}_n = \boldsymbol{v}(x(t^n), t^n),\ p_n = p(x(t^n), t^n),\ \eta_n = \eta(x, t^n),\ \frac{\partial \eta_n}{\partial t} = \frac{\partial \eta}{\partial t}(x, t^n)$, where $t^n = n\Delta t$ and $\eta_0 \equiv \frac{\partial \eta_0}{\partial t} = 0$.

1. Move each point of the grid $x(t^{n+1}) = x(t^n) + s(x(t^{n+1}))$,
   where $s(x(t^{n+1})) = \eta^{(k)}(x, t^{n+1})$ is a given displacement of each grid point $x(t^{n+1}) \in \Omega_{t^{n+1}}^{(k)}$, see [Bro02].

2. Assign the velocity on the interface $\Gamma_{t^{n+1}}^{wall}$ $\quad v_2^{n+1} = v_2^{grid,\ n+1} = \frac{\partial \eta}{\partial t}^{(k)}(x, t^{n+1})$.

3. Solve fluid equations in the new time step t^{n+1} in a given domain $\Omega^{(k)} t^{n+1}$, obtain $\boldsymbol{v}_{n+1}, p_{n+1}$, evaluate the term $\frac{1}{\rho_w h}(p_{n+1} - P_w)$ in the structure equation (5.1).

4. Solve the structure equation (5.1) and obtain a new (k+1)-approximation of deformation
   $\eta^{(k+1)}(x, t^{n+1}),\ \frac{\partial \eta}{\partial t}^{(k+1)}(x, t^{n+1})$ in the new time step (n+1).

5. If $t^{n+1} < T$ put n=n+1 and go to step 1 (a new time step).
   If $t^{n+1} = T$ put $k = k+1$, if $k = K \implies$ STOP, else go to the step 0 (go to the new domain iteration).

Remark 5.1. Our approach differs from the decoupling algorithm given by Formaggia, Gerbeau, Nobile, and Quarteroni in [FGNQ00, page 16]. In their paper, an implicit algorithm for the coupling of fluid and structure equations is used, where fluid and domain are decoupled in each time step. First, the displacement of the structure is extrapolated from the previous time step, then the mesh is updated and the new values of the displacement are obtained from the structure equation in the updated domain. Finally, the new value of the displacement is obtained as a combination of the value from the previous time step and the new, recently computed value. By doing this, a relaxation parameter is used. Moreover, several iterations with respect to the displacement are performed in each time step. This is the sequential approach (see the fixed point method mentioned in [DDFQ06]). The choice of a relaxation parameter is important for the convergence of the method.

We decouple the domain and the fluid globally, i.e. before the time discretisation. Our method requires approximately 3 global iterations, since the values of the deformation will then stabilise, as we show in Section 5.1.3. ◇

5.1.2 On details of the discretisation

Domain deformation equation

We first rewrite the equation governing the upper boundary deformation into a system of two differential equations. Using the notation $\frac{\partial \eta}{\partial t} = \xi$ we obtain the following system:

$$\frac{\partial \xi}{\partial t} - a\frac{\partial^2 \eta}{\partial x^2} + b\eta - c\frac{\partial^2 \xi}{\partial x^2} = \frac{p - P_w}{\rho\rho_w h} \tag{5.2}$$

$$\frac{\partial \eta}{\partial t} = \xi \tag{5.3}$$

We divide the time interval $(0, T)$ to N segments of the length Δt, where $\Delta t = t^n - t^{n-1},\ n = 1, .., N.\ t^0 = 0,\ t^N = T$.

For the time discretisation of the system (5.2)–(5.3), we propose the following scheme

$$\frac{\xi^{n+1} - \xi^n}{\Delta t} - a\alpha\frac{\partial^2 \eta^{n+1}}{\partial x^2} + b\alpha\eta^{n+1} - c\alpha\frac{\partial^2 \xi^{n+1}}{\partial x^2} = \frac{p^n - P_w^n}{\rho\rho_w h} + a(1-\alpha)\frac{\partial^2 \eta^n}{\partial x^2} - b(1-\alpha)\eta^n + c(1-\alpha)\alpha\frac{\partial^2 \xi^n}{\partial x^2}$$

and

$$\frac{\eta^{n+1} - \eta^n}{\Delta t} = \alpha\xi^{n+1} + (1-\alpha)\xi^n,$$

where the parameter $\alpha = 0, \frac{1}{2}, 1$. In this notation, $\xi^{n+1} = \xi(x, t^{n+1})$.

In the sequel we describe the space discretisation of the equations (5.4)–(5.5), where the central difference scheme is used for an approximation of space derivations. We divide the interval $(0, L)$ to M segments of the length $\Delta x = x_i - x_{i-1},\ i = 1, ..M,\ x_0 = 0,\ x_M = L$. With the use of notation $\xi^n_{x_i} = \xi(x_i, t^n)$, our approximation yields

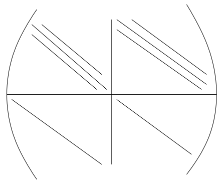
$$\begin{aligned} \frac{\xi^{n+1}_{x_i} - \xi^n_{x_i}}{\Delta t} &- a\alpha \frac{\eta^{n+1}_{x_{i+1}} - 2\eta^{n+1}_{x_i} + \eta^{n+1}_{x_{i-1}}}{\Delta x^2} + b\alpha\eta^{n+1}_{x_i} - c\alpha \frac{\xi^{n+1}_{x_{i+1}} - 2\xi^{n+1}_{x_i} + \xi^{n+1}_{x_{i-1}}}{\Delta x^2} \\ &= \frac{p^n_{x_i} - P^n_{w,\ x_i}}{\rho\rho_w h} + a(1-\alpha)\frac{\eta^n_{x_{i+1}} - 2\eta^n_{x_i} + \eta^n_{x_{i-1}}}{\Delta x^2} \\ &\quad - b(1-\alpha)\eta^n_{x_i} + c(1-\alpha)\frac{\xi^n_{x_{i+1}} - 2\xi^n_{x_i} + \xi^n_{x_{i-1}}}{\Delta x^2}, \end{aligned} \tag{5.4}$$

$$\frac{\eta^{n+1}_{x_i} - \eta^n_{x_i}}{\Delta t} = \alpha\xi^{n+1}_{x_i} + (1-\alpha)\xi^n_{x_i}, \tag{5.5}$$

where the parameter $\alpha = 0, \frac{1}{2}, 1$. For $\alpha = \frac{1}{2}$ we get the Newmark's scheme. For $\alpha = 0$, we obtain the explicit scheme and for $\alpha = 1$ we obtain the implicit scheme.

Remark 5.2 (Numerical aspects). After the discretisation we solve the linear system of equations with unknowns $\xi^n_{x_1}, .., \xi^n_{x_M}, \eta^n_{x_1}, .., \eta^n_{x_M}$ in each time step.

We consider zero Dirichlet boundary condition and zero initial conditions: $\eta^0_{x_i} = \xi^0_{x_i} = \eta^n_{x_0} = \eta^n_{x_M} = 0,\ n = 1, \ldots N,\ i = 0, \ldots M$.



The matrix of this system is block-three-diagonal. A direct method (Gauss elimination) is used to solve this system. ◇

Finite volume method for fluid equations

We first introduce Navier-Stokes equations for moving control volumes, which are derived from Reynolds transport theorem for moving control volumes, see e.g. [Bro02, page 17], or see ALE formulation of fluid flow in a moving domain, [Qua01, Chapter 4]. We divide the computational domain Ω_t into finite volumes $\Omega_{t,i},\ i = 1, \ldots, n_{cv}$, where n_{cv} is the number of control volumes, see Fig. 5.1. The weak formulation for this problem which uses piecewise constant test functions with support in the control volume $\Omega_{t,i},\ i = 1, ..n_{cv}$ is:

$$\frac{\partial}{\partial t}\int_{\Omega_{t,i}} \boldsymbol{v} dV - \int_{\Omega_{t,i}} \mathbf{f} dV$$
$$+ \int_{\partial\Omega_{t,i}} \big(-\mu(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T)\boldsymbol{n}\big) + p\boldsymbol{I}\boldsymbol{n} + (\boldsymbol{v} - \boldsymbol{v}_{grid})\boldsymbol{v}^T\boldsymbol{n} dS = 0$$
$$\int_{\partial\Omega_{t,i}} (\boldsymbol{v} - \boldsymbol{v}_{grid})\boldsymbol{n} = 0$$

The domain movement velocity field $\boldsymbol{v}_{grid}$ is known and it is determined in this case by velocity of the boundary moving $\frac{\partial\eta^{(k)}}{\partial t}$ on Γ_t^w (in $(k+1)$-th domain iteration, see the decoupling algorithm above).

Concerning the time discretisation for the fluid equation (see [Näg03]), the Backward Euler-implicit method is implemented in UG. The time discretisation leads to a nonlinear system. The nonlinear term is linearised in UG in the following way:

$$\boldsymbol{v}\boldsymbol{v}^T\boldsymbol{n} = \tilde{\boldsymbol{v}}\tilde{\boldsymbol{v}}^T\boldsymbol{n} + \tilde{\boldsymbol{v}}^T\boldsymbol{n}(\boldsymbol{v} - \tilde{\boldsymbol{v}}) + \tilde{\boldsymbol{v}}\boldsymbol{n}^T(\boldsymbol{v} - \tilde{\boldsymbol{v}}) + O(\boldsymbol{v} - \tilde{\boldsymbol{v}})^2,$$

where $\tilde{\boldsymbol{v}}$ is some known value of the velocity field, here we use velocity from the previous time-step, $\tilde{\boldsymbol{v}} = \boldsymbol{v}(x, t^{n-1})$ and $\boldsymbol{v} = \boldsymbol{v}(x, t^n)$. If the term $O(\boldsymbol{v} - \tilde{\boldsymbol{v}})^2$ is omitted, then we get a linearisation called Newton's approximation. If both the terms $O(\boldsymbol{v} - \tilde{\boldsymbol{v}})^2$, $\tilde{\boldsymbol{v}}\boldsymbol{n}^T(\boldsymbol{v} - \tilde{\boldsymbol{v}})$ are omitted, then we obtain an approximation called fix-point method.

For the space discretisation, we use the FVM method implemented in UG (see again [Näg03]). The dual finite volume grid is obtained from the finite element grid using a standard technique. The problem is discretised into control volumes

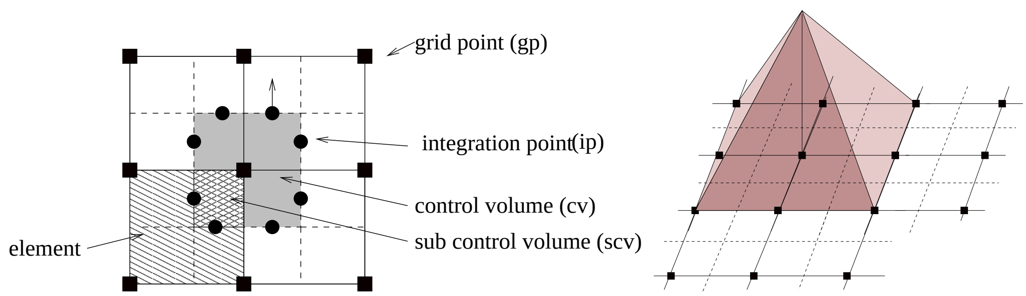


Figure 5.1: Finite volum grid with important points and the nodal basis function $N_k(x)$

using a piecewise-constant test function with support in the control volume $\Omega_{t,i}$, and solved on each control volume. The unknown velocity and pressure are defined in each grid point. The standard piecewise linear nodal basis N_k with support in the neighbouring elements

$$N_k(x_i) = \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{if } k \neq i \end{cases} \qquad i, k = 1, \dots, n_{gp}$$

(where n_{gp} denotes the number of grid points) is used for the numerical approximation of the solution, see Fig. 5.1 right.

In the following approximation of individual terms we omit the time index of the control volume $\Omega_{t,i}$ and we denote it by Ω_i. We denote $\Omega_{i,j}$ the j-th subcontrol volume of Ω_i, see Fig. 5.1 left. We first explain the finite volume approximation.

To approximate the boundary integrals of control volumes, the values are interpolated from the grid points (gp) to the integration points (ip) on the boundary of the control volume, $N_k(ip_j)f(gp_k)$, and then summed over the boundaries of subcontrol volumes (scvb):

$$\sum_{j=1}^{n_{scvb(k)}} \int_{\partial\Omega_{kj}} f(x)dS \approx \sum_{j=1}^{n_{scvb(k)}} |\partial\Omega_{kj}| \sum_{k=1}^{n_{gp}} N_k(ip_j)f(gp_k)$$

The control volume integrals are replaced with the sum over subcontrol volumes (scv) of the grid point values (gp):

$$\sum_{j=1}^{n_{scv(k)}} \int_{\Omega_{kj}} f(x)dV \approx \sum_{j=1}^{n_{scv(k)}} |\Omega_{kj}| f(gp_k)$$

The approximation of individual terms then reads as follows.
Diffusion:

$$\begin{aligned} &-\sum_{k=1}^{n_{gp}} \sum_{j=1}^{n_{scvb}} \int_{\partial\Omega_{kj}} \nu(\nabla \boldsymbol{v} + \nabla^T \boldsymbol{v})\boldsymbol{n} dS \\ &\approx -\sum_{k=1}^{n_{gp}} \sum_{j=1}^{n_{scvb}} \nu(\nabla N_k(ip_j)(\boldsymbol{v} gp_k)) + (\nabla N_k(ip_j)\boldsymbol{v}(gp_k))^T)\boldsymbol{n}_j \end{aligned}$$

Convection:

$$\sum_{k=1}^{n_{gp}} \sum_{j=1}^{n_{scvb}} \int_{\partial\Omega_{k_j}} \boldsymbol{v}\boldsymbol{v}^T \boldsymbol{n} dS \approx \boldsymbol{v}(ip_j)\boldsymbol{v}(ip_j)^T \boldsymbol{n}_j$$

Pressure gradient:

$$\sum_{k=1}^{n_{gp}} \sum_{j=1}^{n_{scvb}} \int_{\partial\Omega_{k_j}} p\boldsymbol{n} dS \approx N_k(ip_j)p(gp_k)\boldsymbol{n}_j$$

Velocity time derivate:

$$\frac{\partial}{\partial t}\sum_{k=1}^{n_{gp}}\sum_{j=1}^{n_{scv}}\int_{\Omega_{kj}}\boldsymbol{v}dV \approx \frac{\partial}{\partial t}\sum_{k=1}^{n_{gp}}\sum_{j=1}^{n_{scv}}|\Omega_{kj}|\boldsymbol{v}(gp_k)$$

Divergence-free condition (continuity equation).

$$\sum_{k=1}^{n_{gp}}\sum_{j=1}^{n_{scvb}}\int_{\partial\Omega_{kj}}\boldsymbol{v}\cdot\boldsymbol{n}dS \approx \sum_{k=1}^{n_{gp}}\sum_{j=1}^{n_{scvb}}\boldsymbol{v}(ip_j)\cdot\boldsymbol{n}_j$$

Here $\boldsymbol{n}_j$ denotes the outward normal vector to the j-th subcontrol volume boundary scaled with $|\partial\Omega_{k,j}|$.

UG stabilisation

We now shortly describe the stabilisation method used in UG (a more detailed explanation can be found in [Näg03]). This method is based on the idea of Raw [Raw85], later modified by Karimian [KS95].

Note that values in integration points $\boldsymbol{v}(ip_j)$ remain undetermined. In order to stabilise the system and to couple the velocity $\boldsymbol{v}(ip_j)$ with the values of the velocity and the pressure on the grid points $\boldsymbol{v}(gp_k)$, we need some relation between values of the velocity in (ip) and velocity and pressure values on (gp) points. Hence the momentum equation is solved in each integration point for each element using the finite difference scheme. The convection term is approximated here using a fix-point linearisation, and the velocity derivate is replaced with the up-wind difference

$$\sum_j v_j\frac{\partial v_i}{\partial x_j} \approx \sum_j \tilde{v}_j\frac{\partial v_i}{\partial x_j}, \quad \frac{\partial v_i}{\partial x_j} \approx \frac{v_i - v_i^{up}}{L}.$$

The up-wind velocity $\boldsymbol{v}^{up}$ can be determined using various techniques which are described in [Näg03]. In this approach, we obtain the following relation

$$\boldsymbol{v}(ip) = \sum_{k=1}^{n_{gp}} C_{\boldsymbol{v}}(gp_k)\cdot\boldsymbol{v}(gp_k) + C_p\frac{\partial p}{\partial x}(ip) + C_t\boldsymbol{v}^{old}, \tag{5.6}$$

where the coefficients $C_{\boldsymbol{v}}$, C_p, C_t represent the relation between the grid point velocity, pressure gradient and the previous time-step velocity respectively. The result of such a stabilisation in UG yields a modification of continuity equation:

$$-\varepsilon\Delta p + \text{ div }(\boldsymbol{v}) = 0, \ \varepsilon \approx h^2,$$

where h is the size of grid element. This result is obtained by replacing the interpolation point values $\boldsymbol{v}(ip)$ occurring in the continuity equation with grid point values according to (5.6).

Remark 5.3 (Numerical aspects in FVM). An implicit time discretisation and linearisation of this nonlinear system yields a system of linear equations. A multi-grid method for solving system of linear equations is implemented in UG. Instructions for programming in UG can be found in [BJR01]. ⋄

5.1.3 Numerical experiments

All numerical experiment in this subsection are related to the problem (2.1)–(2.8). Numerical simulations were also performed using simpler deformation models, see [Zau04]—however, the generalised string model (5.1) has better properties from the mechanical and mathematical point of view. We recall (5.1) with details on physical characteristics listed in Table 5.1.

$$\frac{\partial^2\eta}{\partial t^2} - \frac{\kappa G}{\rho_w}\frac{\partial^2\eta}{\partial x^2} + \frac{h\mathcal{E}}{\rho_w h R_0^2}\eta - \frac{\gamma}{\rho_w h}\frac{\partial^3\eta}{\partial t\partial x^2} = \frac{p-P_w}{\rho\rho_w h}.$$

Two types of boundary conditions on the inflow boundary were used, the Dirichlet type (which prescribes the inflow velocity profile), and the Neumann-type (which prescribes the inflow pressure profile in $\big(\nu(\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^T)-p\mathbf{I}\big)\boldsymbol{n}=P_{in}(t)\big)$. The inflow pressure and inflow velocity are functions of time and radius of the tube (we refer to the planar section of the cylindrical domain as tube). The radius of such a domain corresponds to the vertical variable x_2, see Fig. 5.1.3. Naturally, the inflow pressure profile is more suitable for simulation of the blood flow.

Experimental data	
Inflow *velocity* profile	Inflow *pressure* profile
Kinematic viscosity $\nu = 0.035$	
External pressure acting on the elastic wall $P_w = 0$	
Reference radius $R_0 = 0.5\ cm$	
Tube length $L = 16\ cm,\ L = 8\ cm$	
Time interval $(0,T),\ T = 2\ s$	
Inflow velocity function $g(r,t) = 18(0.25 - r^2).sin^2(\frac{\pi t}{\omega})\ cm.s^{-1}$	Inflow pressure function $P_{in}(r,t) = 100.sin(\frac{2\pi t}{\omega})\ dynes.cm^{-2}$
The half-period $\omega = 1s$	The period $\omega = 1s$
Time step $\Delta t = 0.02\ s$	Time step $\Delta t = 0.005\ s$
Density of the vessel wall tissue $\rho_w = 1.1\ g.cm^{-3}$	
Young's modulus $\mathcal{E} = 0.75 \times 10^4\ dynes.cm^{-2}$	
Timoshenko's shear correction factor $\kappa = 1$	
Shear modulus $G = \frac{\mathcal{E}}{2(1+\delta)},\ \delta = 0.5$ for incompressible materials	
Wall thickness $h = 0.14\ cm$	Wall thickness $h = 0.014\ cm$
Viscoelasticity coefficient $\gamma = 4.10^2$	Viscoelasticity coefficient $\gamma = 4.0$

Table 5.1: Physical constants and other data used in experiments

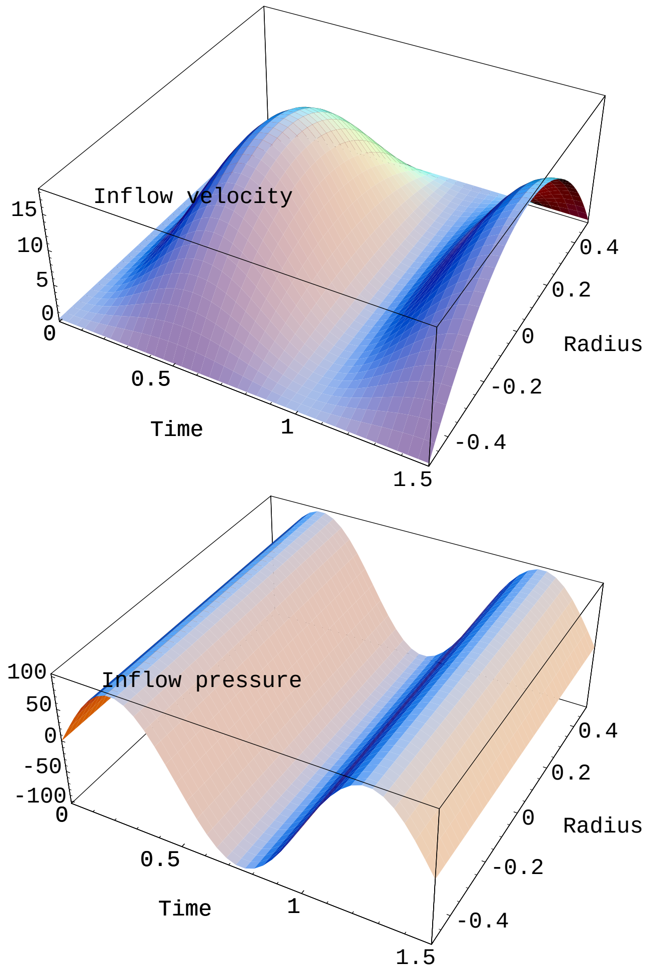


Figure 5.2: Boundary conditions: Inflow velocity and inflow pressure profiles

We use a full implicit time discretisation method with parameter $\alpha = 1$ or semi-implicit scheme ($\alpha = \frac{1}{2}$); and the central difference scheme for the space discretisation (5.4)–(5.5).

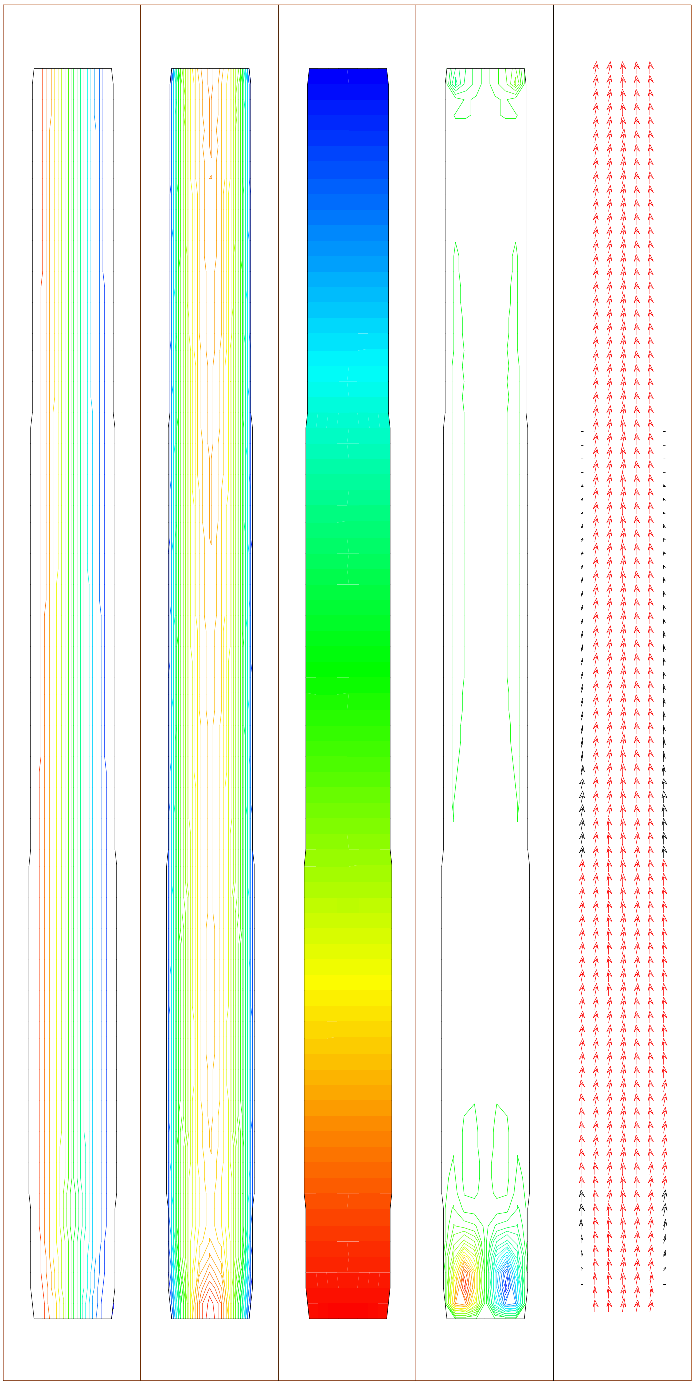


Figure 5.3: Inflow velocity experiment, t=1.3 s. From left to right: streamlines, horizontal velocity, vertical velocity, pressure (UG output)

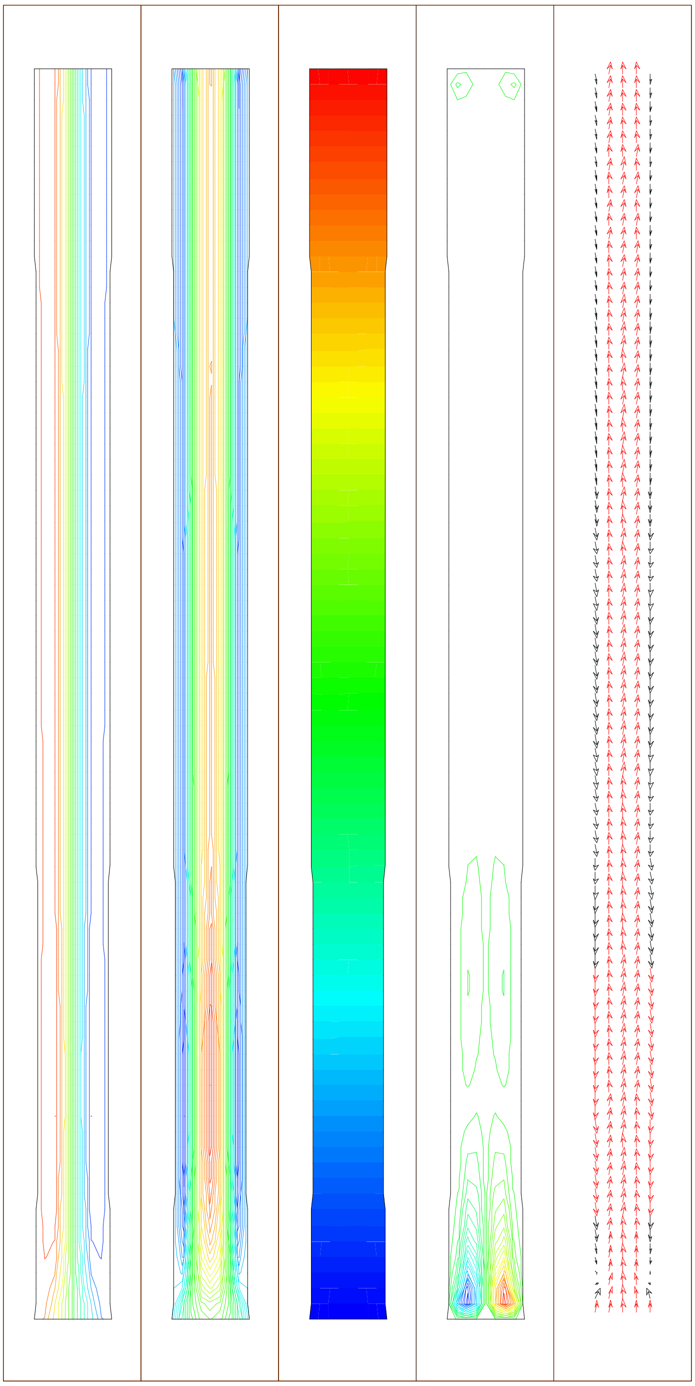


Figure 5.4: Inflow velocity experiment, t=1.86 s. From left to right: streamlines, horizontal velocity, vertical velocity, pressure (UG output)

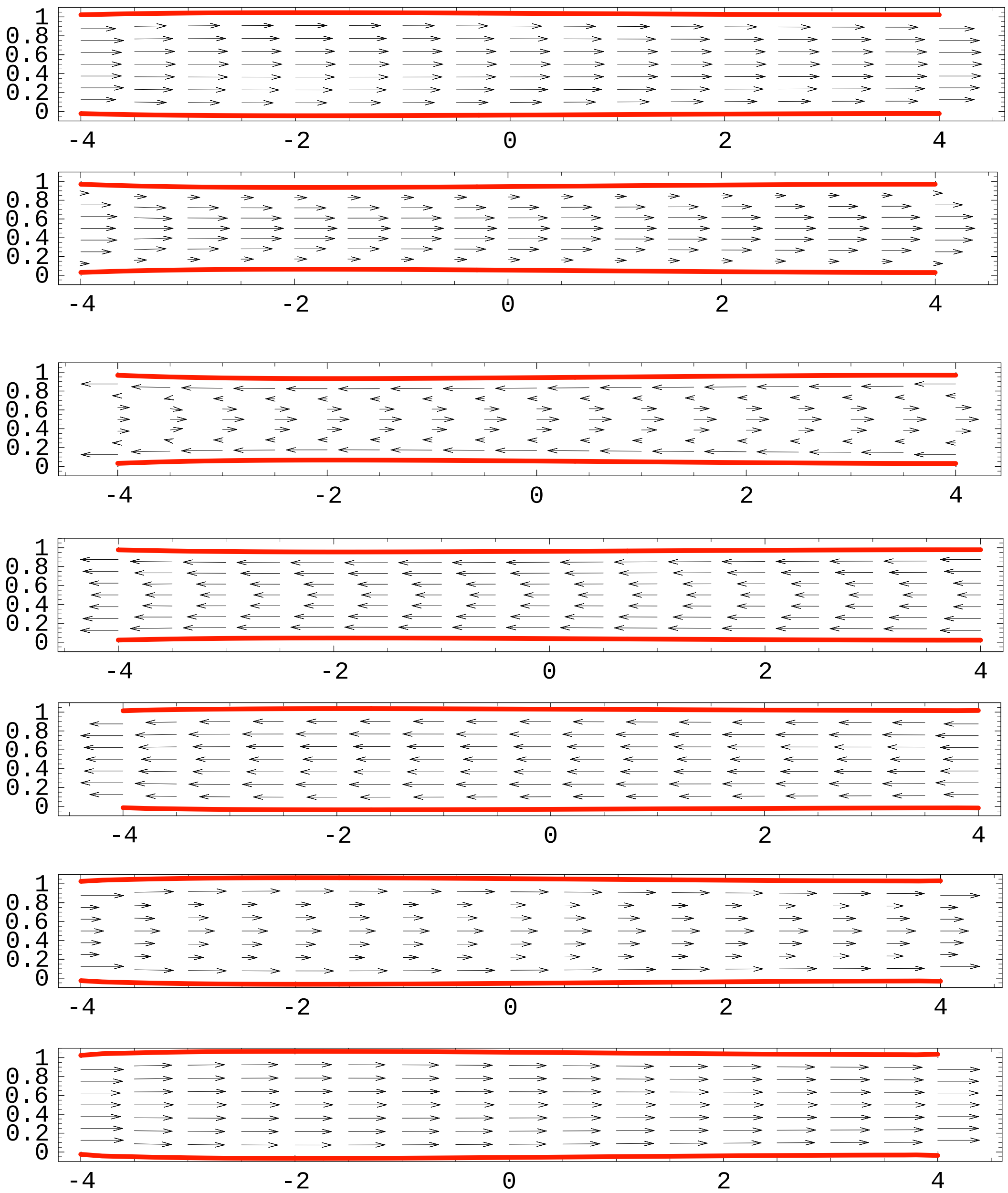


Figure 5.5: Inflow pressure experiment. Velocity field, t=0.4 s, 0.7 s, 0.8 s, 0.9 s, 1.1 s, 1.2 s, 1.3 s from top to bottom (visualisation in Mathematica)

5.1.4 Observations

The changes in the domain deformation caused by the changes of the pressure of the fluid on the deformed boundary are shown in Fig. 5.6. Note that the boundary pressure is contained in the force term (on the right-hand side) of the deformation equation (5.1).

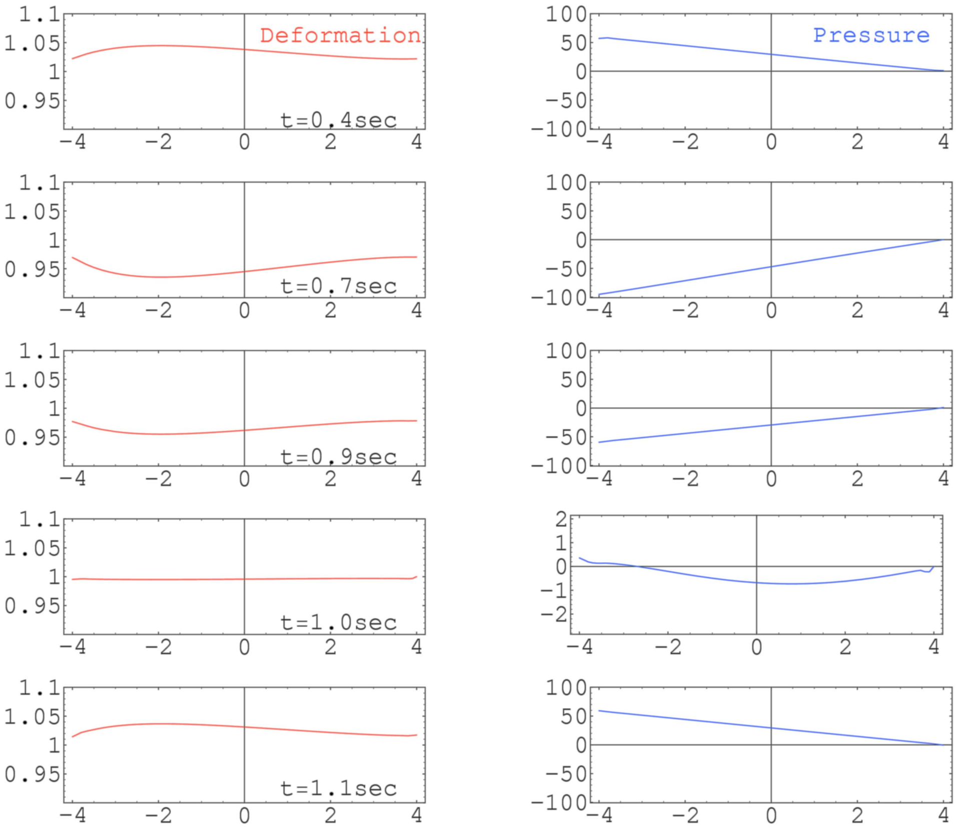


Figure 5.6: Inflow pressure experiment: The dependence of the boundary deformation in cm (left) and the fluid pressure ($g.cm^{-2}$) on the deformed wall (right). For time steps (from top to bottom) t=0.2 s, t=0.4 s, t=0.7 s, t=0.9 s, t=1.0 s, t=1.1 s

Moreover, Fig. 5.7 and Table 5.2 indicate that the domain deformation function converges and stabilises after a few domain 'global' iterations.

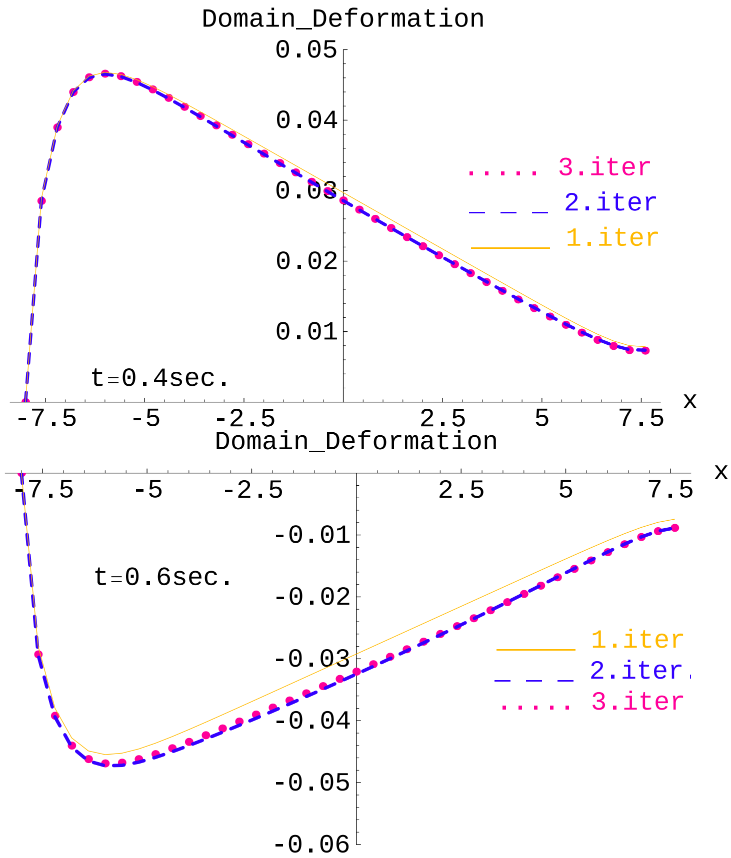


Figure 5.7: Convergence of global iteration with respect to the domain deformation $\eta^{(k)} \rightarrow \eta$ (inflow velocity experiment)

Values of the deformation at time t=0.6 s		
Iteration	Deformation values (in cm) at x=-7.2 cm	x=2.5 cm
1.	-0.037969	-0.021836
2.	-0.039366	-0.024843
3.	-0.039197	-0.024729
5.	-0.040638	-0.020888
9.	-0.040645	-0.020887

Table 5.2: Convergence of the deformation values (y-displacements)

5.2 Our approximated model

In this section, we describe the numerical method for the problem (1.1)–(1.10), where the parameter $\lambda = 1$, see (1.6). Next, note that we proved the existence and uniqueness of the solution to our approximated model only with Laplace operator, see Section 2.3. However, our numerical experiments are performed with the full form of Cauchy stress tensor, see (2.9), as it is implemented in UG.

This approximated problem differs from original Quarteroni's model in the right-hand side of the deformation equation (5.1)

$$H(x,t) = -\frac{\kappa}{E}\left(\frac{\partial \eta}{\partial t}(x,t) - v_2(x,h,t)\right),\ E = \rho_w h.$$

5.2.1 Decoupling

The coupling between the flow and the domain geometry, represented by the interface condition on $\Gamma^{wall}(t)$ (2.10)–(2.11) is approximated by (1.3)–(1.4). The decoupling of the velocity field and the deformation is realised independently on the time discretisation, as we assume that the time development of domain deformation $h(x,t)$ is given and explicitly known. It means, that we apply the global iterative approach from Section 5.1 also to this problem. However, the equation for the fluid flow, and the equation for η remain coupled through conditions (1.3)–(1.4). This coupling is decoupled due to the time discretisation. We describe the decoupling more precisely in the following algorithm.

Global iteration method algorithm

Let K denote the number of domain iterations (given), let T denote the end of the time interval (given), let $k = 0$.

0. Let
   $\Omega(h(x,t)) \equiv \{(x, x_2(t)),\ |x_2(t)| < h(x,t),\ x \in (0,L)\ \ t \in (0,T)\}$ be the given time-dependent computational domain.
   Let $n = 0$, let the initial condition be
   $\boldsymbol{v}_n = \boldsymbol{v}(x(t^n), t^n),\ p_n = p(x(t^n), t^n),\ \eta_n = \eta(x, t^n),\ \frac{\partial \eta_n}{\partial t} = \frac{\partial \eta(x,t^n)}{\partial t}$,
   $(\eta_0 = \frac{\partial \eta_0}{\partial t} \equiv 0)$.

1. Move each point of the grid $x(t^{n+1}) = x(t^n) + s(x(t^{n+1}))$,
   where $s(x(t^{n+1}))$ is a given displacement of each grid point $x(t^{n+1}) \in \Omega(h)$, and it holds on Γ^{wall} that $s(x(t^{n+1})) = h(x, t^{n+1}) - R_0$.

2. Assign the velocity on the interface $\Gamma^{wall}(t^{n+1})$:
   $v_2^{n+1} = v_2^{grid,\ n+1} = \frac{\partial h}{\partial t}(x, t^{n+1})$

3. Solve fluid equations in the new time step t^{n+1} in the given domain $\Omega(h(x,t^{n+1}))$ with the following Neumann-type boundary condition for the second component of the velocity, using values from the last time-step on the right-hand side:

$$\left[\nu\frac{\partial v_2}{\partial x}\left(-\frac{\partial h}{\partial x}\right)+\nu\frac{\partial v_2}{\partial x_2}-p+p_w\right](x,h(x,t^{n+1}),t^{n+1}) \quad (5.7)$$
$$=\left[\frac{1}{2}v_2\left(v_2-\frac{\partial h(x,t^n)}{\partial t}\right)+\kappa\left(\frac{\partial\eta^{(k+1)}(x,t^n)}{\partial t}-v_2\right)\right](x,h(x,t^n),t^n)$$

and obtain $\boldsymbol{v}(x,t^{n+1})$, $(p(x,t^{n+1})$.

4. Evaluate the right-hand side of equation for the deformation

$$-E\left[\frac{\partial^2\eta^{(k+1)}}{\partial t^2}-a\frac{\partial^2\eta^{(k+1)}}{\partial x^2}+b\eta^{(k+1)}-c\frac{\partial^3\eta^{(k+1)}}{\partial t\partial x^2}-\kappa\frac{\partial\eta^{(k+1)}}{\partial t}\right](x,t^{n+1}) \quad (5.8)$$
$$=-\kappa v_2(x,h(x,t^{n+1}),t^{n+1}),$$

solve the structure equation and obtain values of deformation $\eta^{(k+1)}(x,t^{n+1})$, $\frac{\partial\eta}{\partial t}^{(k+1)}(x,t^{n+1})$ at time t^{n+1}.

5. If $t^{n+1}<T$ then put n=n+1 and go to step 1 (new time step).
If $t^{n+1}=T$ then put $h(x,t)=\eta^{(k+1)}(x,t)+R_0$, $k=k+1$.
If $k=K \implies$ then STOP, else go to step 0 (next domain iteration).

5.2.2 On details of discretisation

For the time and space discretisation of Navier-Stokes equation, we use methods implemented in UG (see Section 5.1.2, the finite volume method and the scheme for structure equation). Therefore we omit the details and concentrate on the differences between Quartetoni's and our model. First, the time discretisation of the structure equation, with another term on the right-hand side reads:

$$\frac{\xi^{n+1}-\xi^n}{\Delta t}-a\alpha\frac{\partial^2\eta^{n+1}}{\partial x^2}+b\alpha\eta^{n+1}-c\alpha\frac{\partial^2\xi^{n+1}}{\partial x^2}-\frac{\kappa}{E}\xi^{n+1}$$
$$=-\frac{\kappa}{E}v_2(x,h,t^n)+a(1-\alpha)\frac{\partial^2\eta^{n+1}}{\partial x^2}-b(1-\alpha)\eta^{n+1}+c(1-\alpha)\alpha\frac{\partial^2\xi^{n+1}}{\partial x^2},$$

$$\frac{\eta^{n+1}-\eta^n}{\Delta t}=\alpha\xi^{n+1}+(1-\alpha)\xi^n,$$

where $\xi=\frac{\partial\eta}{\partial t}$, $\alpha=1,\ \frac{1}{2},\ 0$. We denote $\xi^n=\xi(x,t^n)$. For $\alpha=1$ we obtain the implicit scheme, for $\alpha=\frac{1}{2}$ we obtain the Newmark's scheme and for $\alpha=0$ we obtain the explicit scheme.

Second, in the approximation of Neumann-type boundary conditions (1.3), (1.7), (1.8) of Navier-Stokes system we use the following approximations and averaging. The square of the velocity $\frac{\rho}{2}v_i(x,t),\ i=1,2,$ as well as the difference $\frac{\partial\eta}{\partial t}(x,t)-v_2(x,t)\equiv D(x,t)$ are at time $t=t^n$ approximated by values from the previous time step, i.e.:

$$\left[\mu\frac{\partial v_2^n}{\partial x}\left(-\frac{\partial h^n}{\partial x}\right)+\mu\frac{\partial v_2^n}{\partial x_2}-p^n+P_w\right](x,h^n) \tag{5.9}$$

$$=\left[\frac{\rho}{2}v_2^{n-1}\left(v_2^{n-1}-\frac{\partial h^{n-1}}{\partial t}\right)+\kappa\left(\frac{\partial\eta^{n-1}}{\partial t}-v_2^{n-1}\right)\right](x,h^{n-1}) \quad \text{for } \ x\in(0,L),$$

$$\left[\mu\frac{\partial v_1^n}{\partial x}-p^n+P_{out}-\frac{\rho}{2}|v_1^{n-1}|^2\right](L,x_2)=0 \quad \text{for } x_2\in(0,1), \tag{5.10}$$

$$\left[\mu\frac{\partial v_1^n}{\partial x}-p^n+P_{in}-\frac{\rho}{2}|v_1^{n-1}|^2\right](0,x_2)=0 \quad \text{for } x_2\in(0,1), \tag{5.11}$$

In case of the space discretisation, for $x_2\in\Gamma^{out}\cup\Gamma^{in}$, such that $x_i\le x_2<x_{i+1};\ x_i,\ x_{i+1}\in\Gamma^{out}$ or Γ^{in} we consider an averaging of the velocity square in (5.10) and (5.11):

$$\frac{\rho}{2}|v_1^{n-1}|^2(x_i)\approx\frac{\rho}{2}\Big(|v_1|^2(x_i,h(x_i,t^{n-1}),t^{n-1})+|v_1|^2(x_{i+1},h(x_{i+1},t^{n-1}),t^{n-1})\Big).$$

The difference $D(x,t^{n-1})=\frac{\partial\eta(x,t^{n-1})}{\partial t}-v_2(x,t^{n-1})$ in (5.9) is for $x_i<x\le x_{i+1}$ $x_i,\ x_{i+1}\in\Gamma^w$ approximated by:

$$D(x_i,t^{n-1})\approx$$
$$\frac{1}{5}\Big(D(x_{i-2},t^{n-1})+D(x_{i-1},t^{n-1})+D(x_i,t^{n-1})+D(x_{i+1},t^{n-1})+D(x_{i+2},t^{n-1})\Big)$$

We consider corresponding averaging of values at $x_0,\ x_1,\ x_M,\ x_{M-1}$, ($M=\frac{L}{\Delta x}$ is the number of grid-points at the boundary Γ^w.) Finally, in the term $-\frac{\rho}{2}v_2^{n-1}\left(v_2^{n-1}-\frac{\partial h^{n-1}}{\partial t}\right)$ in (5.9) we replace

$$\frac{\partial h(x_i,t^{n-1})}{\partial t}\approx\Upsilon h(x_i,t^{n-1})=\frac{h(x_i,t^{n-1})-h(x_i,t^{n-2})}{\Delta t},$$

and approximate

$$v_2^{n-1}\frac{\partial h^{n-1}}{\partial t}\approx\frac{1}{2}\left(v_2(x_i,t^{n-1})+v_2(x_{i+1},t^{n-1})\right)\frac{1}{2}\left(\Upsilon h(x_i,t^{n-1})+\Upsilon h(x_{i+1},t^{n-1})\right).$$

UG implementation of the Neumann-type boundary condition

The Neumannn-type boundary condition is not a standard part of UG implementation. We combine the OUTFLOW and VDIR_PRS conditions in order to implement it. The OUTFLOW boundary condition means that we prescribe the null deformation stress tensor (normal diffusive flux) for fluid, e.g.

$$(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)\boldsymbol{n} = 0 \text{ on } \partial\Omega.$$

The VDIR_PRS is the Dirichlet condition for the pressure variable, i.e. we put

$$p = P_{\partial\Omega} \text{ on } \partial\Omega.$$

The result of this combination is a new VDIR_PRS condition implemented in the NS problem class library in UG, with moving grid. The relevant part of the listing of UGROOT/ns/pclib/nsfields.c. can be found in Appendix B. We prescribe

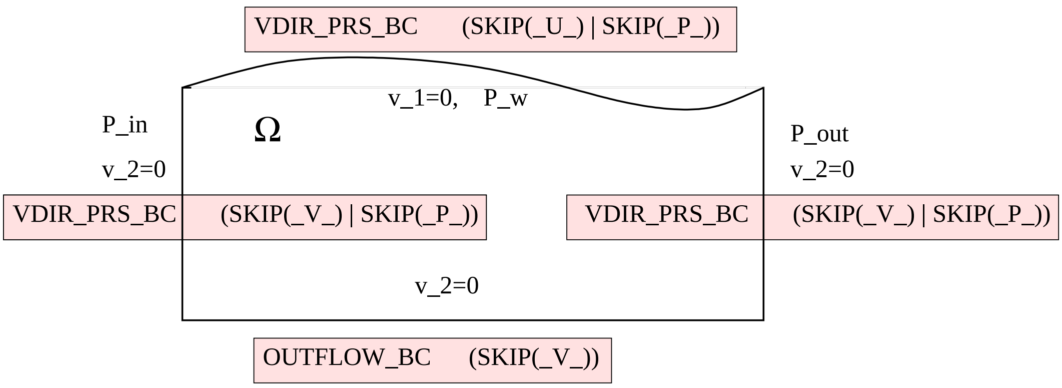


Figure 5.8: Implementation of the boundary condition

the null vertical velocity v_2 (_V_) and the inflow or outflow pressure (_P_) on the boundary $\Gamma^{in}, \Gamma^{out}$ respectively, the null horizontal velocity v_1 (_U_) and the external pressure (_P_) on Γ^w, and the null vertical velocity v_2 (_V_) and the OUTFLOW boundary condition on the symmetry axes Γ^c. According to the approximation of the Neumann-type boundary condition (5.9), (5.10), (5.11) above, we impose the following boundary pressure on $\Gamma^{in}, \Gamma^{out}, \ \Gamma^w$.

$$\begin{aligned}
p(0, x_2, t^n) &= P_{in}(x_2, t^n) - \frac{\rho}{2}|v_1(0, x_2, t^{n-1})|^2 \\
p(L, x_2, t^n) &= P_{out} - \frac{\rho}{2}|v_1(L, x_2, t^{n-1})|^2 \\
p(x, h^n, t^n) &= P_w - \frac{\rho}{2} v_2 \left(v_2 - \Upsilon h^{n-1}\right)(x, h^{n-1}, t^{n-1}) + \kappa \left(v_2 - \frac{\partial \eta}{\partial t}\right)(x, h^{n-1}, t^{n-1})
\end{aligned} \tag{5.12}$$

where the aforementioned averaging of the boundary values is not written here.

5.2.3 Numerical experiments

We performed numerical experiments with the following physical characteristics and data

Experimental data
Kinematic viscosity $\nu = 0.09$
External pressure acting on the elastic wall $P_w = 0$
Reference radius $R_0 = 0.5\ cm$
Tube length $L = 5.1\ cm$
Time interval $(0, T),\ T = 4\ s$
Inflow & outflow pressure functions $P_{in}(r,t) = 6.0 + 10.sin(\frac{2\pi t}{\omega})\ dynes.cm^{-2},\ P_{out} = 0$
The half-period $\omega = 1\ s$
Time step $\Delta t = 0.005\ s$
Density of the vessel wall tissue $\rho_w = 1.1\ g.cm^{-3}$
Young's modulus $\mathcal{E} = 0.75.10^3\ dynes.cm^{-2}$
Timoshenko's shear correction factor $\kappa = 1$
Shear modulus $G = \frac{\mathcal{E}}{2(1+\delta)},\ \delta = 0.5$ for incompressible materials
Wall thickness $h = 0.09\ cm$
Viscoelasticity coefficient $\gamma = 4.0$

Table 5.3: Physical constants and other data used in experiments

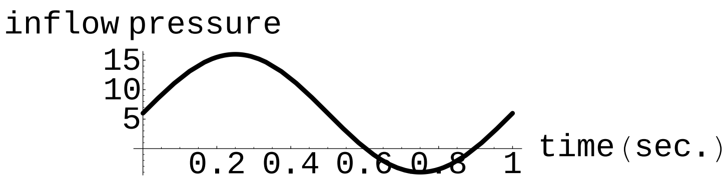


Figure 5.9: Inflow pressure function

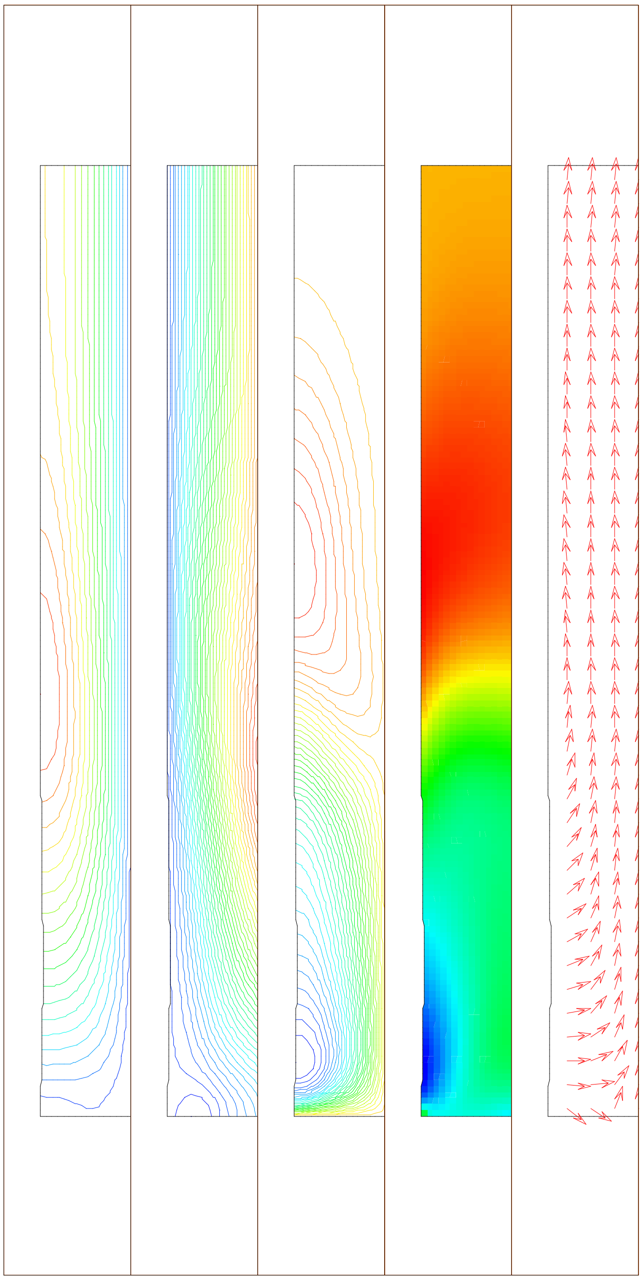


Figure 5.10: UG: Fluid flow in a compliant domain. From left to right: streamlines, horizontal velocity, vertical velocity, pressure and velocity field for $\kappa = 2$, time=3.875 s

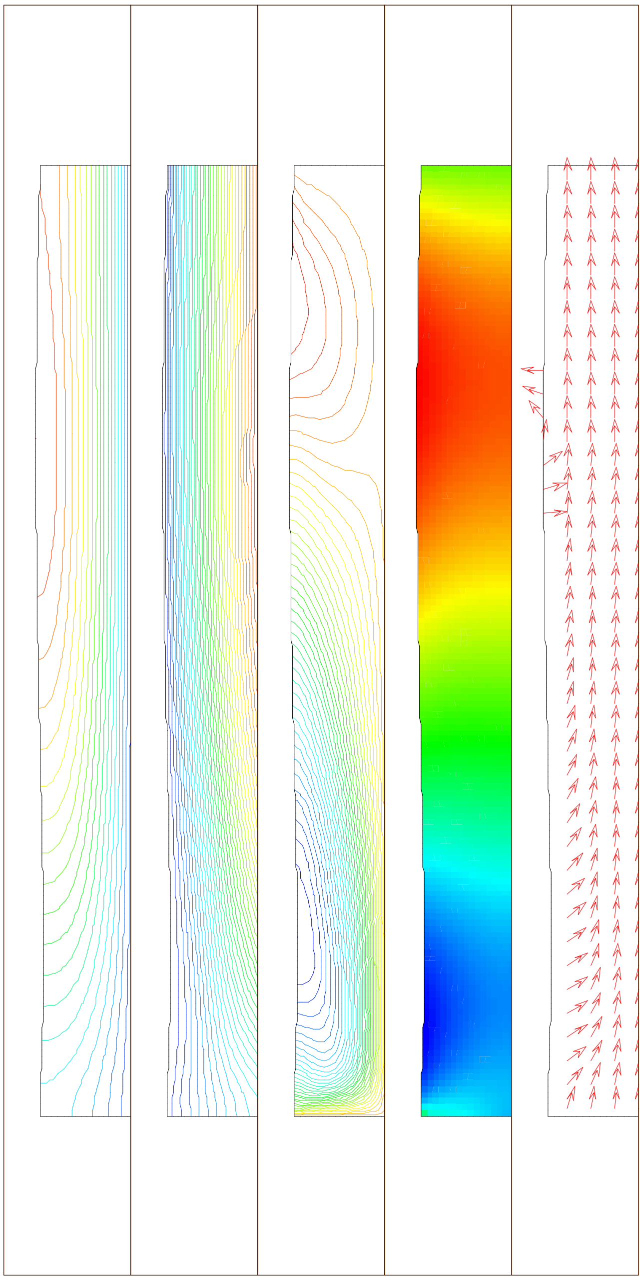


Figure 5.11: UG: Fluid flow in a compliant domain. From left to right: streamlines, horizontal velocity, vertical velocity, pressure and velocity field for $\kappa = 12$, time=3.875 s

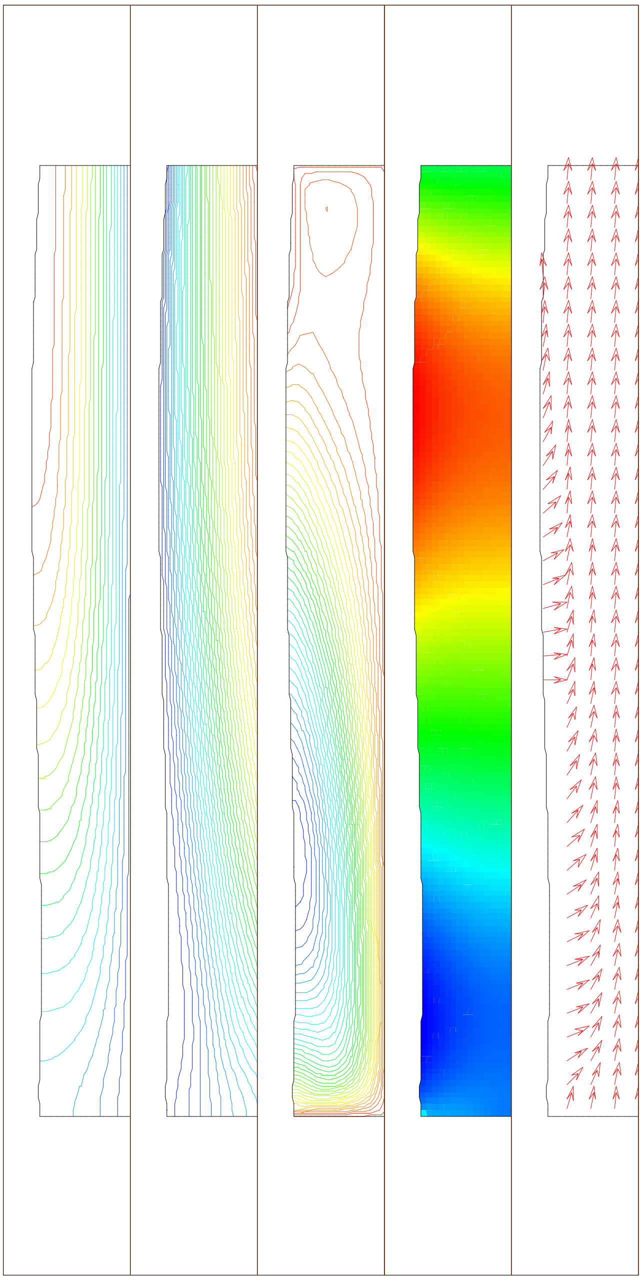


Figure 5.12: UG: Fluid flow in a compliant domain. From left to right: streamlines, horizontal velocity, vertical velocity, pressure and velocity field for $\kappa = 33$, time=3.875 s

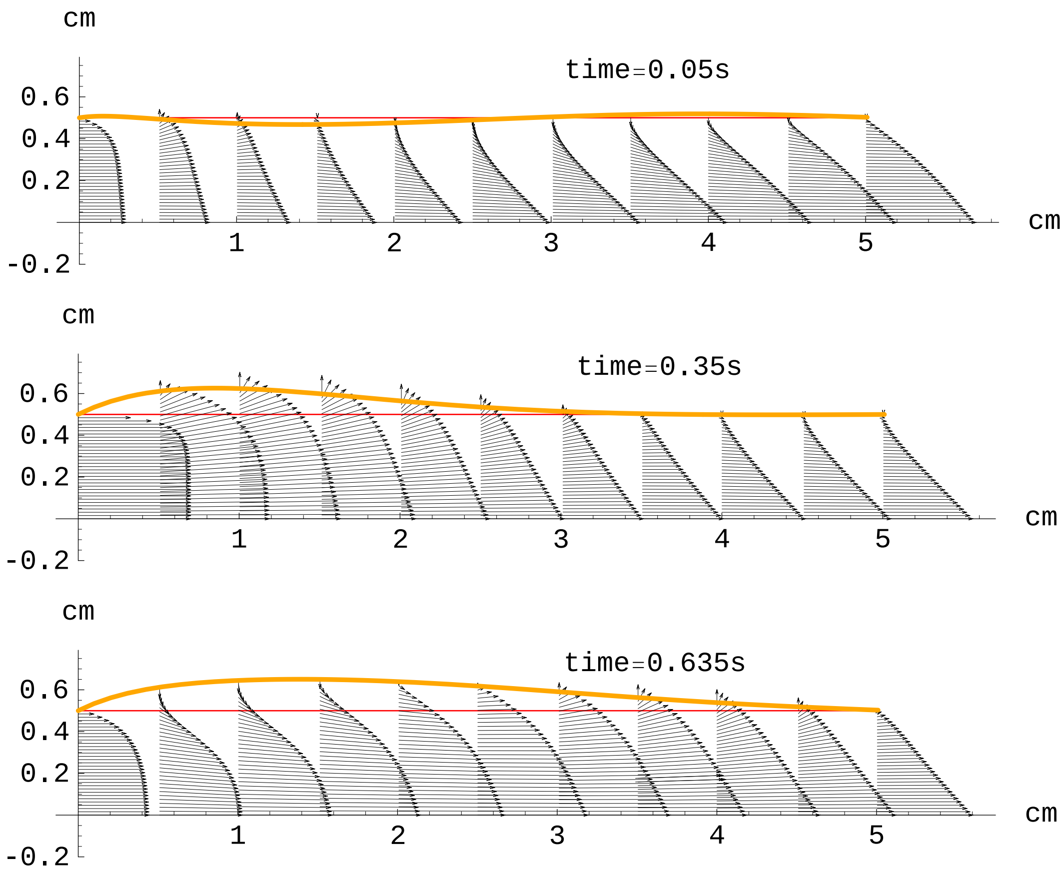


Figure 5.13: Velocity field in a compliant domain (visualisation in Mathematica) for $\kappa = 100$

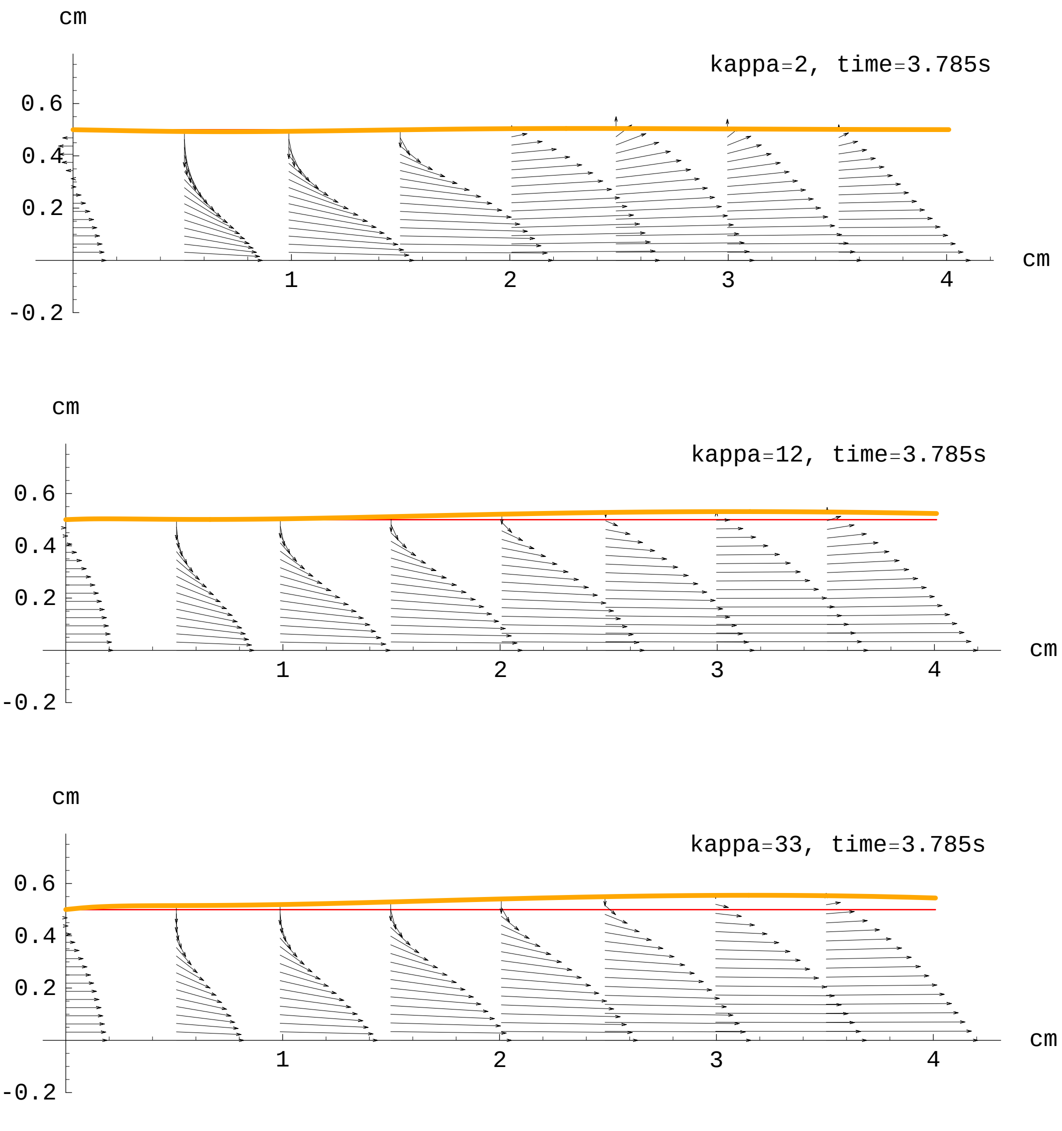


Figure 5.14: Velocity field for varying κ (visualisation in Mathematica) for time=3.875 s

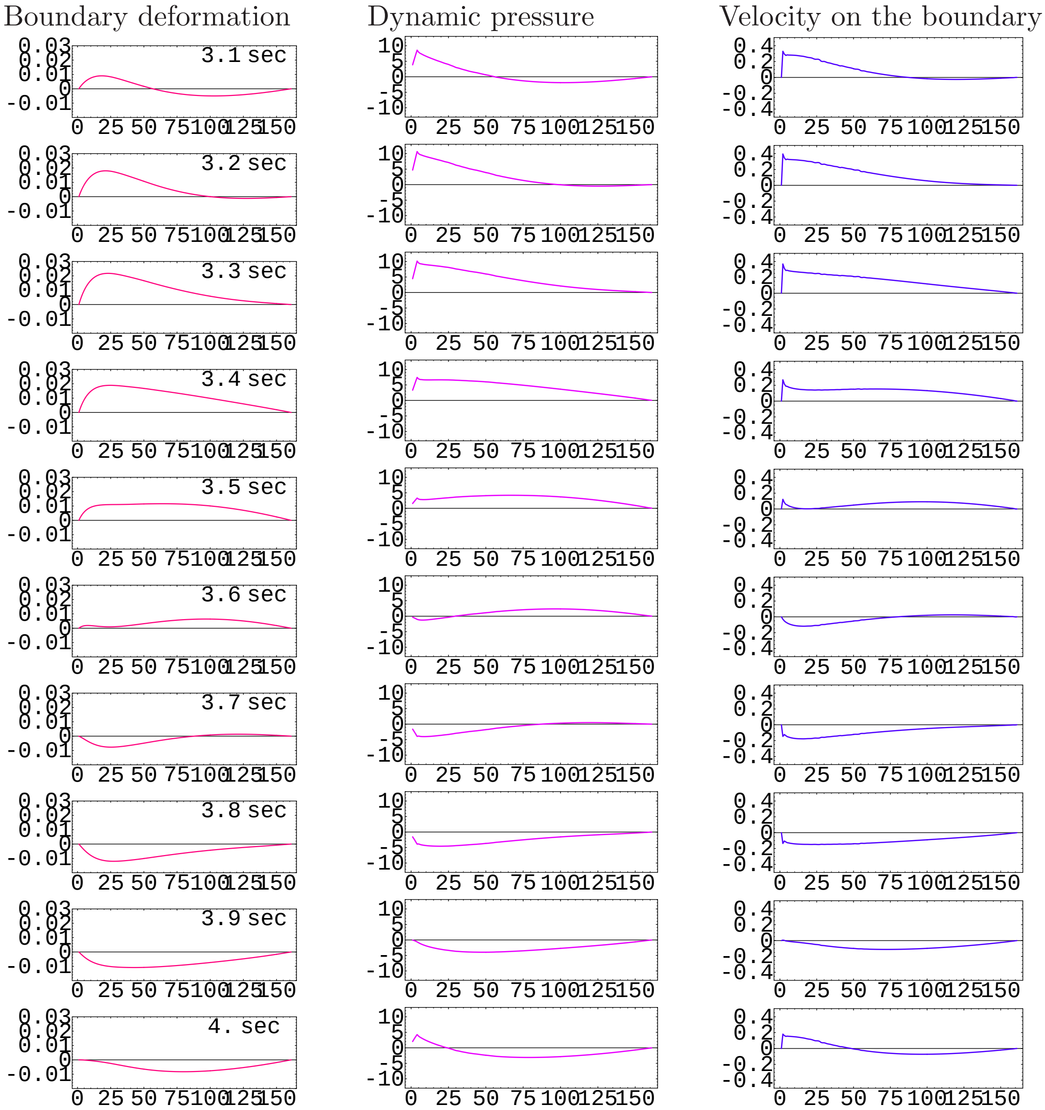


Figure 5.15: Time developement of the boundary deformation, boundary dynamic pressure given by $p|_{\Gamma wall}$ in (5.13) and fluid horizontal velocity on the deformed boundary for $\kappa = 33$, dt=0.005 s, dx=0.03125 cm

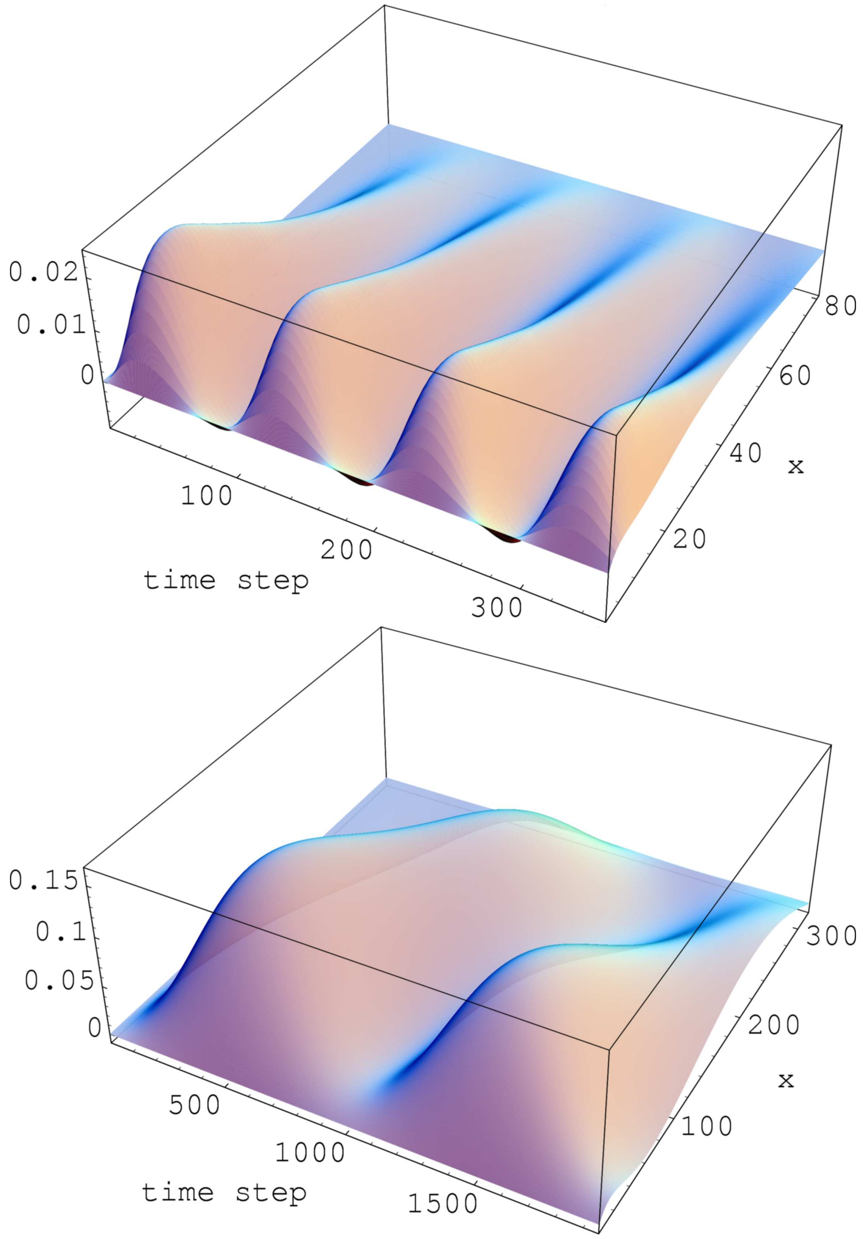


Figure 5.16: Time development of the boundary deformation for $\kappa = 12$ (top) and $\kappa = 100$ (bottom)

5.3 Numerical results

The main goal of this chapter was a simulation of fluid flow in a domain with a deforming part of the boundary, whereby the deformation is dependent on the fluid flow properties. In the first section, we showed some experiments with the original model introduced by Quarteroni, where we used a different method for fluid-domain decoupling. Our experiments indicate that the iterative process for domain geometry converges, see Fig. 5.7 and the Table 5.2.

In the second section of this chapter, we dealed with our approximation of the original problem. We performed some experiments with a fixed parameter κ. Again, a convergence of the domain geometry can be observed, see Fig 5.17 and Table 5.4. It means that the global iteration with respect to the domain deformation is appropriate, although we did not theoretically prove its convergence.

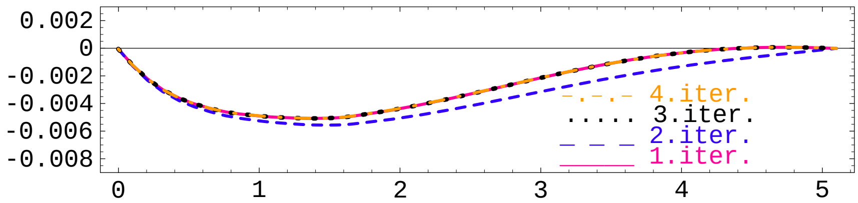


Figure 5.17: Convergence of the wall deformation for $\kappa = 12$, time=2.95 s

Values of the deformation at time t=2.95 s		
Iteration	Deformation values (in cm) for $\kappa = 12$	
	x=1.02 cm	x=3.634 cm
1.	-0.00481124167725	-0.00060430218306
2.	-0.00528328152442	-0.00179846044021
3.	-0.00491893241581	-0.00071200844506
4.	-0.00491116991577	-0.00071094334787
11.	-0.00491118094564	-0.00071051744755

Table 5.4: Convergence of the deformation values (y-displacements)

Another interesting observation is the case of an increasing κ. If $\kappa \to \infty$ then our approximation of the problem of fluid flow in time-dependent domain corresponds to the original problem with a given domain deformation $h(x,t)$ (Remark 4.1). The following figures show experiments with different values of κ. We can see that for increasing κ the domain deformation increases, see Fig. 5.18. Moreover, the fluid flows into the tube and out of the tube through the 'permeable' and deforming boundary Γ^w slower, Fig. 5.19.

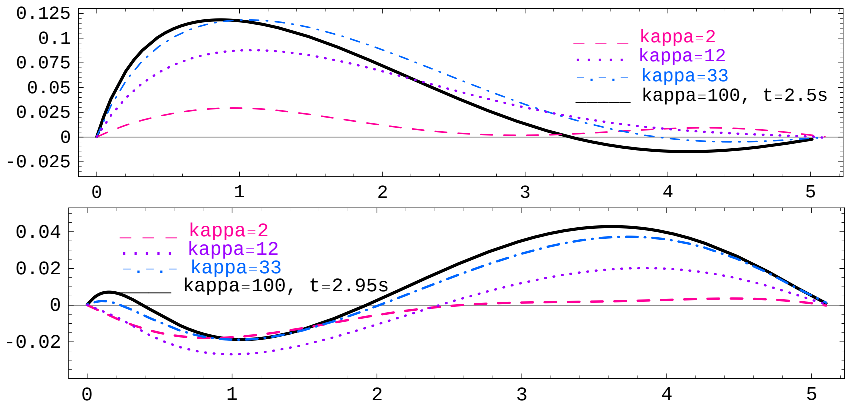


Figure 5.18: Comparison of the moving wall deformations for different κ

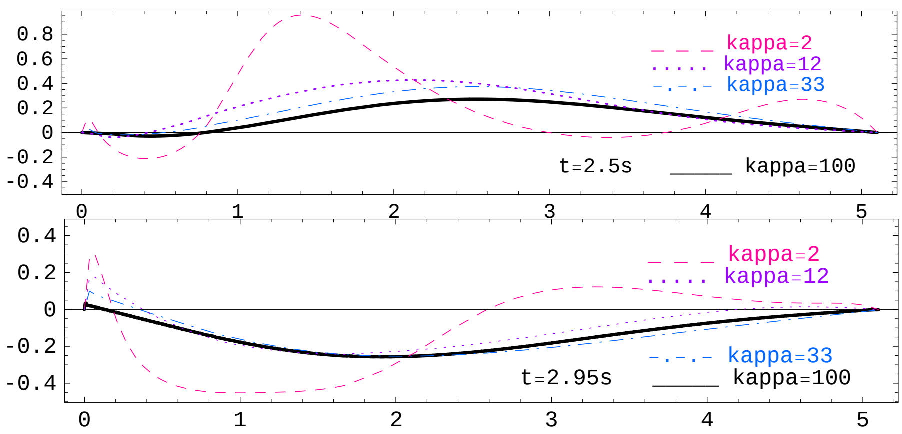


Figure 5.19: Comparison of fluid velocities on the deformed boundary for different κ

Chapter 6

Conclusions

We study the unsteady Navier-Stokes system of equations, which describe the fluid flow in a time-deforming 2D domain. This system of nonlinear partial differential equations is coupled with the equation of the domain deformation on the elastic part of the domain boundary.

In the theoretical part of this thesis, we prove the existence and uniqueness of the weak solution to our perturbed system in which we introduce two approximating parameters ε and κ. In order to get the existence result, we assume an a priori given domain deformation $h(x_1, t)$. Then we transform the perturbed system to a fix domain (rectangle). We use standard techniques in the proof: Rothe's method for the time discretisation, proving the existence of the stationary solution and then deriving a priori estimates, which result into weak convergences of the approximating sequences in the corresponding spaces. The ε-regularisation of the continuity equation is very useful by obtaining the compactness of these sequences in the space L^1 and consequently the strong convergence. Then, by passing to the limit in the weak formulation, we prove the existence of the weak solution. Then we show the uniqueness of this solution and also the continuous dependence on data. In the last step we prove the convergence of solution for $\varepsilon \longrightarrow 0$, i.e. we prove the existence of the solution to the problem introduced in the beginning of this thesis (in Chapter 1).

The numerical part of this thesis deals with a simulation of the pulsating flow in a time-deforming 2D domain, where we use the UG software toolbox with an already implemented support for 2D moving domain. We first present experiments with the problem defined by equations (2.9)–(2.18), which was proposed and studied by Quarteroni et al. Then we deal with a numerical solution for our approximation of the original problem (whose existence and uniqueness is proved in the theoretical part) for physically-based data. We experimentally test the global iterative method for decoupling the unknown domain deformation and the fluid flow. The experiments with Quarteroni's problem as well as our approximated problem indicate the convergence of this method, although we do not prove the convergence theoretically. The domain geometry stabilises after approximately 5 iterations of this global method with around 10^{-5} cm point-wise difference of domain deformations computed in two subsequent iterations.

We also test the influence of the parameter κ to the flow field and the domain geometry. As expected, the amplitude of the domain displacement grows as κ grows and the outflow through the deforming "semi-permeable" part of the boundary decreases. This is a positive result as it suggests a promising κ-approximation of the original problem when $\kappa \longrightarrow \infty$.

Appendix A

Elementary inequalities

The following elementary inequalities can be found in [Eva98, Appendix B: Inequalities]. Let $1 \le p, q \le \infty,\ \frac{1}{p} + \frac{1}{q} = 1,\ u \in L^p(U),\ v \in L^p(U) \cap L^q(U)$,

1. **Young's inequality** $(1 < p, q < \infty)$

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \tag{A.1}$$

$$ab \le \epsilon a^p + C(\epsilon) b^q,\ C(\epsilon) = (\epsilon p)^{-q/p} q^{-1} \quad (a, b > 0,\ \epsilon > 0) \tag{A.2}$$

2. **Hölder's inequality**

$$\int_U |uv| \le \|u\|_{L^p(U)} \|v\|_{L^q(U)} \tag{A.3}$$

3. **Minkowski's inequality**

$$\|u + v\|_{L^p(U)} \le \|u\|_{L^p(U)} + \|v\|_{L^p(U)} \tag{A.4}$$

4. **Discrete Hölder's & discrete Minkowski's inequality**

$$\left| \sum_{k=1}^{n} a_k b_k \right| \le \left(\sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |b_k|^q \right)^{\frac{1}{q}} \tag{A.5}$$

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p \right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^q \right)^{\frac{1}{q}} \tag{A.6}$$

for $a = (a_1, \ldots, a_n),\ b = (b_1, \ldots, b_n) \in R^n,\ 1 \le p < \infty$.

Appendix B

Implementation of Neumann-type boundary condition

```
case VDIR_PRS_BC:
{
  DOUBLE *MyShape = SDV_SHAPEPTR(sdv);
  DOUBLE_VECTOR BIPVel;
  DOUBLE IPweight[MAX_EDGES_OF_CORNER];
  DOUBLE COweight[MAX_CORNERS_OF_ELEM][MAX_CORNERS_OF_SIDE];
  DOUBLE TangentialDiffusiveFlux[MAXNC][DIM * N_UNKNOWN];
  /* NEW */
  DOUBLE_VECTOR normal; /* NEW */
  DOUBLE w;
  INT Nip;
  INT IPs[MAX_EDGES_OF_CORNER];
  INT Nsc;
  INT SCs[MAX_CORNERS_OF_ELEM];
  int j, iip, sco, mpd_co;

  /* interpolate bip-velocity */
  INTPOL_VELOCITY(BIPVel, nco, DD_CONVVELPTRPTR(dd), MyShape);

  /* use coefficients of inner ips with appropriate weights */
  if (GetInnerIPweights(DD_FVG(dd), bip, &Nip, IPs, IPweight))
    REP_ERR_RETURN(ANE_ERROR);

  if (GetInnerCOweights(DD_FVG(dd), bip, &Nsc, SCs, COweight))
    REP_ERR_RETURN(ANE_ERROR);
```

```
V_DIM_COPY(SCVBF_NORMAL(scvbf), normal); /* NEW */
V_DIM_Normalize(normal); /* NEW */

for (iip = 0; iip < Nip; iip++)
{
  ip = IPs[iip];

  if (ComputeDiffusiveFlux(af, ip, Viscosity[ip],
   SCVBF_NORMAL(scvbf), NO,DiffusiveFlux))
    REP_ERR_RETURN(ANE_ERROR);

  for (co = 0; co < nco; co++)
  {
    /* NEW */
    INT j, k;

    for (i = 0; i < DIM; i++)
      for (j = 0; j < N_UNKNOWN; j++)
        TangentialDiffusiveFlux[co][i * N_UNKNOWN + j] =
         DiffusiveFlux[co][i * N_UNKNOWN + j];
    for (i = 0; i < DIM; i++)
      for (j = 0; j < DIM; j++)
        for (k = 0; k < DIM; k++)
          TangentialDiffusiveFlux[co][i*N_UNKNOWN+j] -=
           DiffusiveFlux[co][k * N_UNKNOWN + j] *
           normal[k] * normal[i];
  }

  if (ComputeConvectiveFlux(af, ip, Viscosity[ip],
   SCVBF_NORMAL(scvbf), BIPVel, UpwindShape[ip], delta,
   DD_CONVVELPTRPTR(dd), action, ConvectiveFlux,
   ConvectiveFluxRhs))
    REP_ERR_RETURN (ANE_ERROR);

  if (ComputePressureFlux(af, bip, YES, PressureFlux))
    REP_ERR_RETURN(ANE_ERROR);

  for (co = 0; co < nco; co++)
  {
    /* we map corners of the element to those lying on the
       boundary side */
```

```
    for (sco = 0; sco < Nsc; sco++)
    {
      mpd_co = SCs[sco];
      w = COweight[co][sco] * IPweight[iip];

      /* momentum equation */
      for (i = 0; i < DIM; i++)
      {
        #ifndef V_OUTFLOW
          if (i > _U_)
            break;
        #endif

        for (j = 0; j < DIM; j++)
          DD_MAT(dd, from, mpd_co, i, j) +=
           w * (ConvectiveFlux[co][i * N_UNKNOWN + j] +
           TangentialDiffusiveFlux[co][i * N_UNKNOWN + j]);

        DD_MAT(dd, from, mpd_co, i, _P_) +=
         w * (ConvectiveFlux[co][i * N_UNKNOWN + _P_]);
        #ifdef CD_MAT
          /* uu to ConvDiff */
          DD_DIFF(dd, from, mpd_co) +=
           w * DiffusiveFlux[co][_U_ * N_UNKNOWN + _U_];
          DD_CONV(dd, from, mpd_co) +=
           w * ConvectiveFlux[co][_U_ * N_UNKNOWN + _U_];
        #endif
      }
    }

    /* no corner mapping for pressure */
    for (i = 0; i < DIM; i++)
    {
      #ifndef V_OUTFLOW
        if (i > _U_)
          break;
      #endif

      DD_MAT(dd, from, co, i, _P_) +=
       IPweight[iip] * PressureFlux[co][i];
    }
  }
```

```
    /* rhs contributions */
    for (i = 0; i < DIM; i++)
      DD_RHS(dd, from, i) -= IPweight[iip] *
       ConvectiveFluxRhs[i];
  }

  break;
}
```

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